Computations in homotopy type theory

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Constructivity of Martin–Löf type theory

Theorem
Take $u$ a closed term of type $\mathbb{N}$ in MLTT, and successively apply reduction rules to $u$. Then

- this procedure terminates,
- the order of the reductions does not matter,
- the result is a numeral (of the form $S(\ldots(S(0))\ldots)$).

The last point (canonicity) does not work in the presence of axioms.
Homotopy type theory (HoTT) is

\[ \text{MLTT + Univalence axiom (+ Higher inductive types)} \]

The presence of an axiom destroys the canonicity property. There are closed terms of type \( \mathbb{N} \) which are stuck but are not numerals.
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Nevertheless, univalence “feels” constructive.

**Homotopy canonicity (conjectured by Voevodsky, 2010?)**

Given a closed term \( u : \mathbb{N} \), there exists a closed term \( k : \mathbb{N} \) and a proof \( p : u \equiv_{\mathbb{N}} k \), where \( k \) does not use univalence.
Constructivity of homotopy type theory

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And work in progress towards constructive simplicial models (GH, vdBF).
Many implementations have been written:

- **cubical** (implementation of the first cubical model from BCH)
- **cubicaltt** (cubical type theory from CCHM)
- **redPRL** (cartesian, and in the style of Nuprl)
- **yacctt** (cartesian cubical type theory)
- **redtt** (successor of redPRL)
- **cubical Agda** (based on CCHM and Agda)
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Proposition (2013)

One can construct a natural number $n$ such that

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Open problem
Compute the value of $n$ directly.
(And we tried! But all of our experiments, using the various implementations, ran out of either memory or time.)
The definition

$$\mathbb{Z} \xrightarrow{\lambda n.\text{loop}^n} \Omega S^1 \xrightarrow{\Omega \varphi S^1} \Omega^2 S^2 \xrightarrow{\Omega^2 \varphi S^2} \Omega^3 S^3 \xrightarrow{\Omega^3 e} \Omega^3 (S^1 \ast S^1) \xrightarrow{\Omega^3 \alpha} \Omega^3 S^2$$

$$\Omega^3 (S^1 \ast S^1) \rightarrow \Omega^3 S^3 \rightarrow \Omega^2 \|S^2\|_2 \rightarrow \Omega \|\Omega S^2\|_1 \rightarrow \|\Omega^2 S^2\|_0 \rightarrow \Omega S^1 \rightarrow \mathbb{Z}$$

$n$ is the absolute value of the image of 1