Computations in homotopy type theory

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MLoC 2019, University of Stockholm
August 23, 2019

Constructivity of Martin–Löf type theory

Theorem

Take u a closed term of type $\mathbb{N}$ in MLTT, and successively apply reduction rules to u. Then

- this procedure terminates,
- the order of the reductions does not matter,
- the result is a numeral (of the form $S(\ldots(S(0))\ldots)$).

The last point (canonicity) does not work in the presence of axioms.

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Homotopy type theory (HoTT) is

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Nevertheless, univalence "feels" constructive.

Homotopy canonicity (conjectured by Voevodsky, 2010?)

Given a closed term $u : \mathbb{N}$, there exists a closed term $k : \mathbb{N}$ and a proof $p : u =_{\mathbb{N}} k$, where k does not use univalence.

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And work in progress towards constructive simplicial models (GH, vdBF).

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- → yacctt (cartesian cubical type theory)
- → redtt (successor of redPRL)
- → cubical Agda (based on CCHM and Agda)

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Open problem

Compute the value of n directly.

(And we tried! But all of our experiments, using the various implementations, ran out of either memory or time.)

The definition

$$\mathbb{Z} \xrightarrow{\lambda n.\mathtt{loop}^n} \Omega\mathbb{S}^1 \xrightarrow{\Omega\varphi_{\mathbb{S}^1}} \Omega^2\mathbb{S}^2 \xrightarrow{\Omega^2\varphi_{\mathbb{S}^2}} \Omega^3\mathbb{S}^3 \xrightarrow{\Omega^3 e} \Omega^3(\mathbb{S}^1 * \mathbb{S}^1) \xrightarrow{\Omega^3\alpha} \Omega^3\mathbb{S}^2 \xrightarrow{h}$$

$$\Omega^3(\mathbb{S}^1 * \mathbb{S}^1) \xrightarrow[\Omega^3 e^{-1}]{} \Omega^3\mathbb{S}^3 \xrightarrow[e_3]{} \Omega^2\|\mathbb{S}^2\|_2 \xrightarrow[\Omega\kappa_{2,\mathbb{S}^2}]{} \Omega\|\Omega\mathbb{S}^2\|_1 \xrightarrow[\kappa_{1,\Omega\mathbb{S}^2}]{} \|\Omega^2\mathbb{S}^2\|_0 \xrightarrow[e_2]{} \Omega\mathbb{S}^1 \xrightarrow[e_1]{} \mathbb{Z}$$

n is the absolute value of the image of 1