Towards a constructive simplicial model of Univalent Foundations

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Goal

To define a model of Univalent Foundations that is

(1) definable constructively, i.e. without EM and AC

(2) defined in a category homotopically-equivalent to **Top**.

Univalent Foundations = **ML** + **UA** , where

- **ML** = Martin-Löf type theory with one universe type
- **UA** = Voevodsky's Univalence Axiom

Open problem since ∼2012.

Related work

Cubical approach:

- [BCH], [CCHM], [OP], . . . do (1) but not (2).

Simplicial approach has some advantages:

- more familiar
- uses standard notion of Kan fibration
- straightforward equivalence with **Top**.

Ongoing work:

- [vdBF] attempts both (1) and (2)
- [ACCRS] does (1) and (2) using equivariant fibrations.

Main result

Theorem (Gambino and Henry)**.** Constructively, there exists a comprehension category

$$\begin{array}{ccc} \mathbf{Fib_{cof}} & \xrightarrow{\quad \chi \quad} & \mathbf{SSet}^{\rightarrow}_{\mathbf{cof}} \\ & \searrow \quad \swarrow_{\mathsf{cod}} & \\ & \mathbf{SSet_{cof}} & \end{array}$$

with

- all the type constructors of **ML**
- univalence of the universe
- Π-types are weakly stable, other type constructors are pseudo-stable.

$\mathbf{SSet_{cof}}$ = full subcategory of cofibrant simplicial sets $\subsetneq$ **SSet**

References

[H1] S. Henry
Weak model structures in classical and constructive mathematics
ArXiv, 2018

[H2] S. Henry
A constructive account of the Kan-Quillen model structure and of Kan's Ex^{∞} functor
ArXiv, 2019

[GSS] N. Gambino and K. Szumiło and C. Sattler
The constructive Kan-Quillen model structure: two new proofs
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[GH] N. Gambino and S. Henry
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ArXiv, 2019

Outline of the talk

Part I. Review of the classical simplicial model

Part II. The constructive Kan-Quillen model structure

Part III. Function types and the univalent universe.

Part I

Review of the classical simplicial model

Voevodsky's model

Idea

- contexts = simplicial sets
- dependent types = Kan fibrations.

$\Rightarrow$ The comprehension category

$$\begin{array}{ccc} \mathbf{Fib} & \xrightarrow{\chi} & \mathbf{SSet}^{\to} \\ & \searrow \quad \swarrow_{\mathsf{cod}} & \\ & \mathbf{SSet} & \end{array}$$

It supports

- all the type constructors of **ML**
- a univalent universe

satisfying stability conditions.

It gives rise to a strict model via a splitting process.

Key facts

(0) Existence of the Kan-Quillen model structure on **SSet**.

(1) $A, B \in \mathbf{SSet}$, B Kan complex $\Rightarrow$ B^A Kan complex.

(2) $p : A \to X$ Kan fibration $\Rightarrow$ the right adjoint to pullback

$$\Pi_p : \mathbf{SSet}_{/A} \to \mathbf{SSet}_{/X}$$

preserves Kan fibrations.

(3) There is a Kan fibration $\pi : \tilde{U} \to U$, with U Kan complex, that classifies small Kan fibrations, i.e.

$$\begin{array}{ccc} A & \longrightarrow & \tilde{U} \\ {\scriptstyle\forall}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \underset{\exists}{\longrightarrow} & U \end{array}$$

(4) The Kan fibration $\pi : \tilde{U} \to U$ is univalent.

Constructivity problems

- Kan-Quillen model structure has classical proofs.
- [BCP] shows that (1), (2) require classical logic.
- [GS] fixed (1), (2) by introducing **uniform** Kan fibrations in **SSet**, but this creates problems for (3), (4).

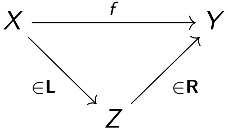
Part II

The constructive Kan-Quillen model structure

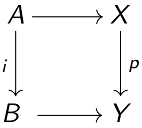
Quick review of model structures (I)

A **weak factorisation system** on a category $\mathcal{E}$ consists of two classes of maps $(\mathbf{L}, \mathbf{R})$ closed under retracts and such that

- For every $f : X \to Y$ there is



- For every $i \in \mathbf{L}$, $p \in \mathbf{R}$, and diagram



  there is a diagonal filler $j : B \to X$.

Examples.

- In **Set**, (decidable) injections and (split) surjections.
- In **MLTT**, identity types [GG]

Quick review of model structures (II)

A **model structure** on a category $\mathcal{E}$ consists of three classes of maps

$$(\mathbf{W}, \mathbf{C}, \mathbf{F})$$

such that

- $\mathbf{W}$ satisfies 3-for-2
- $(\mathbf{C}, \mathbf{W} \cap \mathbf{F})$ is a weak factorisation system,
- $(\mathbf{W} \cap \mathbf{C}, \mathbf{F})$ is a weak factorisation system.

Idea. The wfs $(\mathbf{W} \cap \mathbf{C}, \mathbf{F})$ is used to interpret identity types [AW]

Example. In **Gpd**, there is the model structure of categorical equivalences, functors injective on objects and isofibrations (cf. Hofmann-Streicher).

Constructive simplicial homotopy theory

We start with

$$I = \{ \partial\Delta_n \to \Delta_n \mid n \geq 0 \}$$

$$J = \{ \Lambda_n^k \to \Delta_n \mid 0 \leq k \leq n \}$$

and generate wfs's

- $(\mathbf{Sat}(I), I^{\pitchfork})$ = 'cofibrations' and 'trivial fibrations'
- $(\mathbf{Sat}(J), J^{\pitchfork})$ = 'trivial cofibrations' and 'fibrations'.

We wish to have a model structure $(\mathbf{W}, \mathbf{C}, \mathbf{F})$ such that

$$\mathbf{C} = \mathbf{Sat}(I), \qquad \mathbf{W} \cap \mathbf{F} = I^{\pitchfork}$$

$$\mathbf{W} \cap \mathbf{C} = \mathbf{Sat}(J), \qquad \mathbf{F} = J^{\pitchfork}$$

In particular, $\mathbf{F}$ = Kan fibrations. This helps with (3).

Constructive cofibrations

Let $\mathbf{C} = \mathbf{Sat}(I)$.

Classically, for $i : A \to B$ in **SSet**, TFAE

- $i \in \mathbf{C}$
- i is a monomorphism

Constructively, for $i : A \to B$ in **SSet**, TFAE

- $i \in \mathbf{C}$
- i is a monomorphism s.t. $\forall n$, $i_n : A_n \to B_n$ is complemented, i.e.

$$\forall y \in B_n \big(y \in A_n \vee y \notin A_n \big) \,,$$

  and degeneracy of simplices in $B_n \setminus A_n$ is decidable.

Note. $\mathbf{C}$ = cofibrations in Reedy wfs generated by the wfs

(Complemented mono, Split epi)

on **Set**.

The constructive Kan-Quillen model structure

Theorem [H2]**.** Constructively, the category **SSet** admits a model structure $(\mathbf{W}, \mathbf{C}, \mathbf{F})$ such that

$$\mathbf{C} = \mathbf{Sat}(I)\,, \quad \mathbf{F} = \text{ Kan fibrations }.$$

Two other proofs in [GSS].

Note

- Constructively, not every object is cofibrant: X is cofibrant if and only if degeneracy of simplices in X is decidable.
- Every object X has a cofibrant replacement, given by $\mathbb{L}(X)$ cofibrant and $t : \mathbb{L}(X) \to X$ in $\mathbf{W} \cap \mathbf{F}$.

Outline of one proof from [GSS]

This is inspired by [GS] and [S].

1. Construct the wfs's as above.
2. Prove a restricted Frobenius property: in a pullback square

$$\begin{array}{ccc} B & \longrightarrow & A \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ Y & \longrightarrow & X \end{array}$$

   with $Y \to X$ a fibration with Y cofibrant, if $i \in \mathbf{Sat}(J)$ then $j \in \mathbf{Sat}(J)$.
3. Prove the equivalence extension property in $\mathbf{SSet_{cof}}$
4. Establish a model structure on $\mathbf{SSet_{cof}}$
5. Extend it to **SSet** via cofibrant replacement functor.

Part III

Function types and the univalent universe

Towards a constructive simplicial model

Idea

- use cofibrancy to solve constructivity issues,
- contexts are cofibrant simplicial sets,
- types are Kan fibrations between cofibrant simplicial sets.

$\Rightarrow$ The comprehension category

$$\begin{array}{ccc} \mathbf{Fib}_{\mathbf{cof}} & \xrightarrow{\quad \chi \quad} & \mathbf{SSet}^{\to}_{\mathbf{cof}} \\ & \searrow \quad \swarrow_{\mathrm{cod}} & \\ & \mathbf{SSet}_{\mathbf{cof}} & \end{array}$$

Challenge

- stay within the cofibrant fragment.

Key facts

0. Existence of the constructive Kan-Quillen model structure.
1. $A, B \in \mathbf{SSet}$, A cofibrant, B Kan $\Rightarrow B^A$ Kan.
2. $p : A \to X$ Kan fibration, A cofibrant $\Rightarrow$ the right adjoint to pullback

$$\Pi_p : \mathbf{SSet}_{/A} \to \mathbf{SSet}_{/X}$$

   preserves Kan fibrations.

3. There is a Kan fibration $\pi_c : \tilde{U}_c \to U_c$, with U_c cofibrant Kan complex, that weakly classifies small Kan fibrations between cofibrant simplicial sets

$$\begin{array}{ccc} A & \longrightarrow & \tilde{U}_c \\ {\scriptstyle \forall}\downarrow & & \downarrow{\scriptstyle \pi_c} \\ X & \xrightarrow[\exists]{} & U_c \end{array}$$

4. The fibration $\pi_c : \tilde{U}_c \to U_c$ is univalent.

Function types

Let A, B be cofibrant Kan complexes.

Step 1. Consider B^A, which is a Kan complex by (1). We have

$$\mathsf{app}\colon B^A \times A \to B$$

universal, i.e. such that

$$\frac{X \xrightarrow{\;f\;} B^A}{X \times A \xrightarrow{f \times 1_A} B^A \times A \xrightarrow{\mathsf{app}} B}$$

is a bijection. Its inverse is written

$$\frac{X \times A \xrightarrow{\;b\;} B}{X \xrightarrow{\lambda(b)} B^A}$$

In general, B^A is not cofibrant.

Step 2. Let $\mathbb{L}(B^A)$ be a cofibrant replacement of B^A, with

$$t : \mathbb{L}(B^A) \to B^A \quad \text{in} \quad \mathbf{W} \cap \mathbf{F}$$

Now $\mathbb{L}(B^A)$ is cofibrant Kan complex. We have

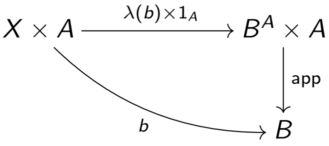
$$\widetilde{\mathsf{app}} \colon \mathbb{L}(B^A) \times A \xrightarrow{t \times 1_A} B^A \times A \xrightarrow{\mathsf{app}} B$$

For $b : X \times A \to B$, with X cofibrant Kan complex, we get

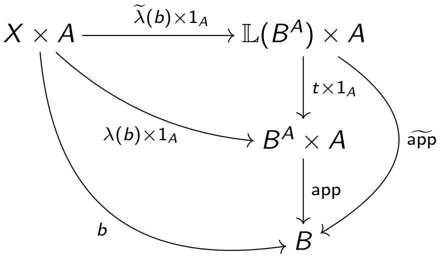
$$\frac{\dfrac{X \times A \xrightarrow{b} B}{X \xrightarrow{\lambda(b)} B^A}}{X \xrightarrow{\widetilde{\lambda}(b)} \mathbb{L}(B^A)} \qquad \text{where} \qquad \begin{array}{ccc} 0 & \longrightarrow & \mathbb{L}(B^A) \\ \downarrow & \overset{\widetilde{\lambda}(b)}{\nearrow} & \downarrow t \\ X & \xrightarrow[\lambda(b)]{} & B^A \end{array}$$

Judgemental β-rule

For standard exponential



For its cofibrant replacement



Propositional η-rule

For $f : X \to \mathbb{L}(B^A)$, we have a homotopy

$$\eta_f : f \sim \widetilde{\lambda}(\widetilde{\mathsf{app}}(f \times 1_A))\,,$$

by

$$\begin{array}{ccc}
\partial\Delta[1] \times X & \xrightarrow{\left[f, \widetilde{\lambda}\left(\widetilde{\mathsf{app}}(f \times 1_A)\right)\right]} & \mathbb{L}(B^A) \\
\downarrow & \nearrow & \downarrow t \\
\Delta[1] \times X & \longrightarrow & B^A\,,
\end{array}$$

where bottom map is given by η-rule for standard exponential.

The universe (I)

Step 1. Construct a Kan fibration $\pi : \tilde{U} \to U$ which classifies small Kan fibrations with cofibrant fibers.

$$U_n = \{p : A \to \Delta[n] \mid p \text{ small fibration}, A \text{ cofibrant}\}$$

Step 2.

- Let $U_c = \mathbb{L}(U)$ be the cofibrant replacement of U, with $t : U_c \to U$ in $\mathbf{W} \cap \mathbf{F}$
- Pullback

$$\begin{array}{ccc} \tilde{U}_c & \longrightarrow & \tilde{U} \\ {\scriptstyle \pi_c}\downarrow & & \downarrow{\scriptstyle \pi} \\ U_c & \xrightarrow[t]{} & U \end{array}$$

The universe (II)

Proposition. The map $\pi_c : \tilde{U}_c \to U_c$ classifies small Kan fibrations between cofibrant objects.

Proof. Let $p : A \to X$ be such a map. Since p has cofibrant fibers, we have

$$\begin{array}{ccc} A & \longrightarrow & \tilde{U} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X & \underset{a}{\longrightarrow} & U \end{array}$$

But

$$\begin{array}{ccc} & & U_c \\ & \overset{a_c}{\nearrow} & \big\downarrow{\scriptstyle t} \\ X & \underset{a}{\longrightarrow} & U \end{array}$$

and so

$$\begin{array}{ccccc} A & \longrightarrow & \tilde{U}_c & \longrightarrow & \tilde{U} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi_c} & & \big\downarrow{\scriptstyle \pi} \\ X & \underset{a_c}{\longrightarrow} & U_c & \underset{t}{\longrightarrow} & U\,. \end{array}$$

□

Fibrancy and univalence of the universe

Step 1. Prove equivalence extension property.

- **Key Lemma.** Let $f : Y \to X$ be a cofibration between cofibrant objects. If $q : B \to Y$ has cofibrant domain, then so does $\Pi_f(q) : \Pi_Y(B) \to X$.

Step 2. Prove U Kan complex, so that U_c is a cofibrant Kan complex.

- Familiar argument, via instance of equivalence extensional property.

Step 3. Prove π univalent, so that π_c univalent.

- Equivalence extension property
- Diagram-chasing, using 3-for-2 for **W**.

Coherence issues

The comprehension category

$$\begin{array}{ccc} \mathbf{Fib_{cof}} & \xrightarrow{\quad\chi\quad} & \mathbf{SSet}^{\rightarrow}_{\mathbf{cof}} \\ & \searrow \qquad \swarrow_{\mathrm{cod}} & \\ & \mathbf{SSet_{cof}} & \end{array}$$

It is not split and satisfies only weak versions of stability conditions.

Open problem. Can we construct a strict model from this?

None of the known strictification methods seems to apply constructively.

Future work

- Solve coherence problem.
- Generalise from **Set** to a Grothendieck topos $\mathcal{E}$
  - Model structure on simplicial sheaves $[\Delta^{\mathrm{op}}, \mathcal{E}]$
  - Connections to higher topos theory
- A simplicial type theory, extracted from the comprehension category, in which univalence axiom is provable.

References

[H1] S. Henry
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