A Unifying Cartesian Cubical Set Model

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Homotopy type theory and univalent foundations

Aims at providing a *practical* foundations for mathematics built on type theory

Started by Vladimir Voevodsky around 2006 and is being actively developed in various proof assistants (AGDA, COQ, LEAN, ...)

Allows *synthetic* reasoning about spaces and homotopy theory as well as new approaches for formalizing (higher) abstract mathematics

These foundations are compatible with classical logic

Homotopy type theory and univalent foundations

$$\text{Univalent Type Theory} = \text{MLTT} + \text{Univalence}$$
$$\text{Homotopy Type Theory} = \text{UTT} + \text{Higher Inductive Types}$$

Theorem (Voevodsky, Kapulkin-Lumsdaine)

Univalent Type Theory has a model in Kan simplicial sets

Problem: inherently classical, how to make this constructive?

Cubical Methods

Breakthrough, using cubical methods:

Theorem (Bezem-Coquand-Huber, 2013)

Univalent Type Theory has a constructive model in "substructural" Kan cubical sets ("BCH model").

This led to development of a variety of cubical set models

$$\widehat{\square} = [\square^{\mathrm{op}}, \mathbf{Set}]$$

Cubical Methods

Inspired by BCH we constructed a model based on "structural" cubical sets with connections and reversals:

Theorem (Cohen-Coquand-Huber-M., 2015)

Univalent Type Theory has a constructive model in De Morgan Kan cubical sets ("CCHM model").

We also developed a **cubical type theory** in which we can prove and compute with the **univalence theorem**

$$\mathsf{ua} : (A\ B : \mathcal{U}) \to (\mathsf{Path}_{\mathcal{U}}\ A\ B) \simeq (A \simeq B)$$

Cubical Methods

In parallel with the developments in Sweden many people at CMU were working on models based on *cartesian* cubical sets

The crucial idea for constructing univalent universes in cartesian cubical sets was found by Angiuli, Favonia, and Harper (AFH, 2017) when working on computational cartesian cubical type theory. This then led to:

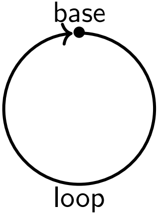
Theorem (Angiuli-Brunerie-Coquand-Favonia-Harper-Licata, 2017)

Univalent Type Theory has a constructive model in cartesian Kan cubical sets ("ABCFHL model").

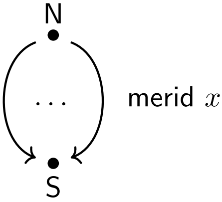
Higher inductive types (HITs)

Types generated by point and path constructors:

$\mathbb{S}^1$:



$\Sigma\mathbb{S}^1$:



These types are added axiomatically to HoTT and justified[1] semantically in Kan simplicial sets (Lumsdaine-Shulman, 2017)

[1] Modulo issues with universes...

Higher inductive types

The cubical set models also support[2] HITs:

- CCHM style cubes: Coquand-Huber-M. (2018)
- Cartesian cubes: Cavallo-Harper (2018)
- BCH: as far as I know not known even for $\mathbb{S}^1$, problems related to Path$(A) := \mathbb{I} \multimap A$

[2] Without universe issues.

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In summary: we get many cubical set models of HoTT

This work: how are these cubical set models related?

[2] Without universe issues.

Cubical Type Theory

What makes a type theory "cubical"?

Add a formal interval $\mathbb{I}$:

$$r, s \quad ::= \quad 0 \mid 1 \mid i$$

Extend the contexts to include interval variables:

$$\Gamma \quad ::= \quad \bullet \mid \Gamma, x : A \mid \Gamma, i : \mathbb{I}$$

Proof theory

$$\frac{\Gamma, i : \mathbb{I} \vdash \mathcal{J}}{\Gamma \vdash \mathcal{J}(\epsilon/i)} \text{ FACE}$$

$$\frac{\Gamma \vdash \mathcal{J}}{\Gamma, i : \mathbb{I} \vdash \mathcal{J}} \text{ WEAKENING}$$

$$\frac{\Gamma, i : \mathbb{I}, j : \mathbb{I} \vdash \mathcal{J}}{\Gamma, j : \mathbb{I}, i : \mathbb{I} \vdash \mathcal{J}} \text{ EXCHANGE}$$

$$\frac{\Gamma, i : \mathbb{I}, j : \mathbb{I} \vdash \mathcal{J}}{\Gamma, i : \mathbb{I} \vdash \mathcal{J}(j/i)} \text{ CONTRACTION}$$

Semantics

$$\Gamma \xrightarrow{d^i_\epsilon} \Gamma, i : \mathbb{I}$$

$$\Gamma, i : \mathbb{I} \xrightarrow{\sigma_i} \Gamma$$

$$\Gamma, j : \mathbb{I}, i : \mathbb{I} \xrightarrow{\tau_{i,j}} \Gamma, i : \mathbb{I}, j : \mathbb{I}$$

$$\Gamma, i : \mathbb{I} \xrightarrow{\delta_{i,j}} \Gamma, i : \mathbb{I}, j : \mathbb{I}$$

Cubical Type Theory

All cubical set models have face maps, degeneracies and symmetries

BCH does not have contraction/diagonals, making it substructural

The cartesian models have contraction/diagonals, making them a good basis for cubical type theory

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We can also consider additional structure on $\mathbb{I}$:

$$r, s \quad ::= \quad 0 \mid 1 \mid i \mid r \wedge s \mid r \vee s \mid \neg r$$

Axioms: connection algebra (OP model), distributive lattice (Dedekind model), De Morgan algebra (CCHM model), Boolean algebra...

Varieties of Cubical Sets - Buchholtz, Morehouse (2017)

Kan operations / fibrations

To get a model of HoTT/UF we also need to equip all types with **Kan operations**: any open box can be filled

Kan operations / fibrations

To get a model of HoTT/UF we also need to equip all types with **Kan operations**: any open box can be filled

Given $(r, s) \in \mathbb{I} \times \mathbb{I}$ we add operations:

$$\frac{\begin{array}{c}\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash s : \mathbb{I} \\ \Gamma \vdash \varphi : \Phi \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash u_0 : A(r/i)[\varphi \mapsto u(r/i)]\end{array}}{\Gamma \vdash \mathsf{com}_i^{r \to s}\, A\, [\varphi \mapsto u]\, u_0 : A(s/i)[\varphi \mapsto u(s/i), (r = s) \mapsto u_0]}$$

Semantically this corresponds to fibration structures

The choice of which (r, s) to include varies between the different models

Cube shapes / generating cofibrations

Another parameter: which shapes of open boxes are allowed (Φ)

Semantically this corresponds to specifying the generating cofibrations, typically these are classified by maps into Φ where Φ is taken to be a subobject of Ω

The crucial idea for supporting univalent universes in AFH was to include "*diagonal cofibrations*" – semantically this corresponds to including $\Delta_{\mathbb{I}} : \mathbb{I} \to \mathbb{I} \times \mathbb{I}$ as a generating cofibration

Cubical set models of HoTT/UF

	Structural	$\mathbb{I}$ operations	Kan operations	Diag. cofib.
BCH			$0 \to r$, $1 \to r$	
CCHM	✓	$\wedge$, $\vee$, $\neg$ (DM alg.)	$0 \to 1$	
Dedekind	✓	$\wedge$, $\vee$ (dist. lattice)	$0 \to 1$, $1 \to 0$	
OP	✓	$\wedge$, $\vee$ (conn. alg.)	$0 \to 1$, $1 \to 0$	
AFH, ABCFHL	✓		$r \to s$	✓

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AFH, ABCFHL	✓		$r \to s$	✓

This work: cartesian cubical set model without diagonal cofibrations

Key idea: don't require the $(r = s)$ condition in com strictly, but only up to a path

Cubical set models of HoTT/UF

Question: which of these cubical set models give rise to model structures where the fibrations correspond to the Kan operations?

Cubical set models of HoTT/UF

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Theorem (Sattler, 2017): constructive model structure using ideas from the cubical models for CCHM, Dedekind and OP models

Theorem (Coquand-Sattler, Awodey): model structure for cartesian cubical sets based on AFH/ABCFHL fibrations with diagonal cofibrations

This work: generalize this to the setting without connections and diagonal cofibrations

Orton-Pitts internal language model

We present our model in the internal language of $\widehat{\square}$ following

Axioms for Modelling Cubical Type Theory in a Topos
Orton, Pitts (2017)

We also formalize it in AGDA and for univalent universes we rely on[3]

Internal Universes in Models of Homotopy Type Theory
Licata, Orton, Pitts, Spitters (2018)

[3] Disclaimer: only on paper so far, not yet formalized.

Orton-Pitts style internal language model

In fact, none of the constructions rely on the subobject classifier $\Omega : \widehat{\square}$, so we work in the internal language of a LCCC $\mathcal{C}$ and do the following:

1. Add an interval $\mathbb{I}$
2. Add a type of cofibrant propositions Φ
3. Define fibration structures
4. Prove that fibration structures are closed under Π, Σ and Path
5. Define univalent fibrant universes of fibrant types
6. Prove that this gives rise to a Quillen model structure

(Parts of the last 2 steps are not yet internal in our paper)

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The interval $\mathbb{I}$

The axiomatization begin with an interval type

$$\mathbb{I} : \mathcal{U} \qquad\qquad 0 : \mathbb{I} \qquad\qquad 1 : \mathbb{I}$$

satisfying

$$\begin{aligned}
\mathbf{ax}_1 &: (P : \mathbb{I} \to \mathcal{U}) \to ((i : \mathbb{I}) \to P\ i \uplus \neg(P\ i)) \to \\
&\qquad ((i : \mathbb{I}) \to P\ i) \uplus ((i : \mathbb{I}) \to \neg(P\ i)) \\
\mathbf{ax}_2 &: \neg(0 = 1)
\end{aligned}$$

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Cofibrant propositions

We also assume a universe à la Tarski of generating cofibrant propositions

$$\Phi : \mathcal{U} \qquad\qquad [\,\_\,] : \Phi \to \texttt{Prop}$$

with operations

$$(\_ \approx 0) : \mathbb{I} \to \Phi \qquad\qquad \vee : \Phi \to \Phi \to \Phi$$
$$(\_ \approx 1) : \mathbb{I} \to \Phi \qquad\qquad \forall : (\mathbb{I} \to \Phi) \to \Phi$$

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$$(\_ \approx 1) : \mathbb{I} \to \Phi \qquad\qquad \forall : (\mathbb{I} \to \Phi) \to \Phi$$

satisfying

$$\mathbf{ax}_3 : (i : \mathbb{I}) \to [\,(i \approx 0)\,] = (i = 0)$$

$$\mathbf{ax}_4 : (i : \mathbb{I}) \to [\,(i \approx 1)\,] = (i = 1)$$

$$\mathbf{ax}_5 : (\varphi\,\psi : \Phi) \to [\,\varphi \vee \psi\,] = [\,\varphi\,] \vee [\,\psi\,]$$

$$\mathbf{ax}_6 : (\varphi : \Phi)\,(A : [\,\varphi\,] \to \mathcal{U})\,(B : \mathcal{U})\,(s : (u : [\,\varphi\,]) \to A\ u \cong B) \to$$
$$\Sigma(B' : \mathcal{U}), \Sigma(s' : B' \cong B), (u : [\,\varphi\,]) \to (A\ u, s\ u) = (B', s')$$

$$\mathbf{ax}_7 : (\varphi : \mathbb{I} \to \Phi) \to [\,\forall\varphi\,] = (i : \mathbb{I}) \to [\,\varphi\, i\,]$$

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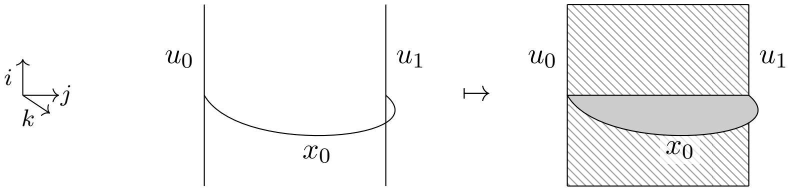
Example: weak composition

Given u_0 and u_1 at $(j \approx 0)$ and $(j \approx 1)$ together with x_0 at $(i \approx r)$, the weak composition and path from r to i is



Weak fibration structures

Given $r : \mathbb{I}$, $A : \mathbb{I} \to \mathcal{U}$, $\varphi : \Phi$, $f : [\,\varphi\,] \to \mathsf{Path}(A)$ and $x_0 : (A\ r)[\varphi \mapsto f \cdot i]$, a *weak composition structure* is given by two operations

$$\mathsf{wcom} : (s : \mathbb{I}) \to (A\ s)[\varphi \mapsto f \cdot s]$$
$$\underline{\mathsf{wcom}} : \mathsf{fst}\,(\mathsf{wcom}\ r) \sim \mathsf{fst}\,x_0$$

satisfying $(i : \mathbb{I}) \to f \cdot r \nearrow \underline{\mathsf{wcom}}\ i$.

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satisfying $(i : \mathbb{I}) \to f \cdot r \nearrow \underline{\mathsf{wcom}}\ i$.

A *weak fibration* (A, α) over $\Gamma : \mathcal{U}$ is a family $A : \Gamma \to \mathcal{U}$ equipped with

$$\begin{aligned} \mathsf{isFib}\ A \triangleq (r : \mathbb{I})\ (p : \mathbb{I} \to \Gamma)\ (\varphi : \Phi)\ (f : [\,\varphi\,] \to (i : \mathbb{I}) \to A(p\ i)) \\ (x_0 : A(p\ r)[\varphi \mapsto f \cdot r]) \to \mathsf{WComp}\ r\ (A\ \circ\ p)\ \varphi\ f\ x_0 \end{aligned}$$

Orton-Pitts style internal language model

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A model of HoTT/UF based on weak fibrations

Using $\mathbf{ax}_1 - \mathbf{ax}_5$ we can prove that isFib is closed under Σ, Π, Path and that natural numbers are fibrant if $\mathcal{C}$ has a NNO

The proofs are straightforward adaptations of the AFH/ABCFHL proofs, but extra care has to be taken to compensate for the weakness

Semantically closure of isFib under Π corresponds to the "Frobenius" property (pullback along fibrations preserve trivial cofibrations)

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A model of HoTT/UF based on weak fibrations

Following Orton-Pitts we can use $\mathbf{ax}_6$ to define Glue types and using $\mathbf{ax}_7$ we can prove that they are also fibrant (by far the most complicated part)

Semantically this corresponds to the "Equivalence Extension Property": equivalences between fibrations extend along cofibrations

Theorem (Universe construction, LOPS)

If $\mathbb{I}$ is tiny, then we can construct a universe U *with a fibration* El *that is classifying in the sense of LOPS Theorem 5.2.*

A model of HoTT/UF based on weak fibrations

We hence get a model of HoTT/UF based on cartesian cubical sets with weak fibrations, without using diagonal cofibrations

What is the relationship to the other models?

AFH fibrations

Inspired by AFH and ABCFHL we can define

$$\begin{aligned}\text{isAFHFib } A \triangleq\ & (r : \mathbb{I})(p : \mathbb{I} \to \Gamma)(\varphi : \Phi)(f : [\,\varphi\,] \to (i : \mathbb{I}) \to A(p\ i))\\ & (x_0 : A(p\ r)[\varphi \mapsto f \cdot r]) \to \text{AFHComp } r\ (A\ \circ\ p)\ \varphi\ f\ x_0\end{aligned}$$

If we assume diagonal cofibrations

$$\begin{aligned}(\_ \approx \_) &: \mathbb{I} \to \mathbb{I} \to \Phi\\ \mathbf{ax}_\Delta &: (r\ s : \mathbb{I}) \to [\,(r \approx s)\,] = (r = s)\end{aligned}$$

then we can prove

Theorem

Given $\Gamma : \mathcal{U}$ *and* $A : \Gamma \to \mathcal{U}$*, we have* isAFHFib A *iff we have* isFib A*.*

CCHM fibrations

Inspired by OP we can define:

$$\text{isCCHMFib } A \triangleq (\varepsilon : \{0,1\})(p : \mathbb{I} \to \Gamma)(\varphi : \Phi)(f : [\,\varphi\,] \to (i : \mathbb{I}) \to A(p\ i))$$
$$(x_0 : A(p\ \varepsilon)[\varphi \mapsto f \cdot r]) \to \mathsf{CCHMComp}\ \varepsilon\ (A\ \circ\ p)\ \varphi\ f\ x_0$$

If we assume a *connection algebra*

$$\begin{aligned}
&\sqcap, \sqcup : \mathbb{I} \to \mathbb{I} \to \mathbb{I} \\
&\mathbf{ax}_{\sqcap} : (r : \mathbb{I}) \to (0 \sqcap r = 0 = r \sqcap 0) \wedge (1 \sqcap r = r = r \sqcap 1) \\
&\mathbf{ax}_{\sqcup} : (r : \mathbb{I}) \to (0 \sqcup r = r = r \sqcup 0) \wedge (1 \sqcup r = 1 = r \sqcup 1)
\end{aligned}$$

then we can prove

Theorem

Given $\Gamma : \mathcal{U}$ *and* $A : \Gamma \to \mathcal{U}$, *we have* isCCHMFib A *iff we have* isFib A.

A model of HoTT/UF based on weak fibrations

This hence generalizes the structural cubical set models (AFH/ABCFHL, CCHM, OP, Dedekind...)

We have also proved that this gives rise to a Quillen model structure

There are 3 parts involved in this:

1. Cofibration - Trivial Fibration awfs
2. Trivial Cofibration - Fibration awfs
3. 2-out-of-3 for weak equivalences

Cofibration-trivial fibration awfs

Cofibrant propositions $[\,-\,] : \Phi \to \texttt{Prop}$ correspond to a monomorphism

$$\top \colon \Phi_{\mathsf{true}} \rightarrowtail \Phi$$

where $\Phi_{\mathsf{true}} \triangleq \Sigma(\varphi : \Phi), [\,\varphi\,] = 1$

Definition (Generating cofibrations)

Let $m \colon A \to B$ be a map in $\mathcal{C}$. We say that m is a *generating cofibration* if it is a pullback of $\top$.

Get (C, F^t) awfs by a version of the small object argument

Trivial cofibration-fibration awfs

Theorem (Weak fibrations and fibrations)

f is a weak fibration iff it has the fibred right lifting property against the map $L_{\mathbb{I}\times\Phi}(\Delta) \,\hat{\times}_{\mathbb{I}\times\Phi} \top$ in $\mathcal{C}/(\mathbb{I}\times\Phi)$

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We say that $m : A \to B$ has the *weak left lifting property* against $f : X \to Y$ if there is a diagonal map as in

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "m"'] & X \arrow[d, "f"] \\
B \arrow[r, "b"] \arrow[ur, dotted] \arrow[u, phantom, "\sim" near end] & Y
\end{tikzcd}
$$

Theorem (Weak fibrations and weak LLP)

f is a weak fibration iff for every object B, every map $r : 1_B \to \mathbb{I}_B$ and generating cofibration $m : A \to B$ in $\mathcal{C}$, r has the weak left lifting property against $\widehat{\mathrm{hom}}_B(B^*(m), f)$.

A model structure based on weak fibrations

We now adapt Sattler's theorem in order to obtain a full model structure.

Theorem (Model structure)

Suppose that $\mathcal{C}$ satisfies axioms $\mathbf{ax}_1$–$\mathbf{ax}_5$ and that every fibration is U*-small for some universe of small fibrations where the underlying object* U *is fibrant. Let (C, F^t) and (C^t, F) be the awfs defined above, then C and F form the cofibrations and fibrations of a model structure on $\mathcal{C}$.*

Theorem

The class C^t is as small as possible subject to

1. *For every object B, the map $\delta_{B0} : B \to B \times \mathbb{I}$ belongs to C^t.*
2. *C and C^t form the cofibrations and trivial cofibrations of a model structure.*

Model structure summary

This model hence also gives rise to a model structure

What is the relationship to the existing model structures constructed from cubical set models of HoTT?

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As the (co)fibrations coincide with the ones in the other model structures we recover them when assuming appropriate additional structure (diagonal cofibrations for cartesian and connections for Dedekind)

Summary

We have:

- Constructed a model of HoTT/UF that generalizes the earlier cubical set models, except for the BCH model
- Mostly formalized in AGDA
- Adapted Sattler's model structure construction to this setting

Future work:

- Formalize the universe construction and model structure in AGDA-$\flat$
- What about BCH? Is it inherently different or does it fit into this generalization?
- Relationship between model structures and the standard one on Kan simplicial sets? Can we also incorporate the equivariant model?

Thank you for your attention!

https://github.com/mortberg/gen-cart/blob/master/conference-paper.pdf