Direct families of sets and direct spectra of Bishop spaces

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Our aim is

to use fundamental notions of BST, an informal theory that somehow complements Bishop's theory of sets, in the theory of Bishop spaces, a function-theoretic approach to constructive topology.

Motivation I

- The theory of sets underlying (BISH) was only sketched in Chapter 3 of [1]. Since Bishop's central aim in [1] was to show that a large part of advanced mathematics can be done within a constructive and computational framework that does not contradict the classical practice, the inclusion of a detailed account of the set-theoretic foundations of BISH could be against the effective delivery of his message (P. Halmos: Naive Set Theory, 1960).
- The BCMT in [5], was very different from the BMT in [1], and the inclusion of an enriched version of the former into [6], affected the corresponding Chapter 3 in two main respects. First, the inductively defined notion of the set of Borel sets generated by a given family of complemented subsets of a set X was excluded, as unnecessary, and, second, the operations on the complemented subsets of a set X were defined differently, and in accordance to the needs of the new measure theory.

Yet, in both books many issues were left untouched, a fact that often was a source of confusion.

Motivation II

- In BCMT $\mathcal{P}(X)$ was treated as a set, while in BMT Bishop generally avoided the powerset by using appropriate families of subsets instead (Zeuner). In later works of Bridges and Richman [7] [9], $\mathcal{P}(X)$ was clearly used as a set, partially in contrast to the predicative spirit of [1].
- The concept of a family of sets indexed by a (discrete) set, was asked to be defined in [1] (Exercise 2, p. 72), and a definition, attributed to Richman, was given in [6] (Exercise 2, p. 78). An elaborate study though, of this concept within BISH is missing, despite its extensive use in BMT and in constructive algebra.
- In [9] Richman introduced the more general notion of a family of objects of a category indexed by some set, but the categorical component in the resulting mixture of Bishop's set theory and category theory was not explained in constructive terms, as it was done in the the formulation of category theory in HoTT (Chapter 9 in [13]).

Motivation III

- Bishop discussed the formal aspects of BISH in [4], where Σ, a variant of Gödel's T, was proposed as a formal system for it. There he also sketched very briefly the implementation of Σ into Algol (Buffalo-meeting, August 1968) .
- In his unpublished work [2] he introduced a version of dependent type theory with one universe to formalise BISH. He also wrote an unpublished paper [3] on the implementation of his type theory into Algol. I think that this is done after Σ.

Bi67, p. 65 and BB85, p. 70

an element u of $\bigcap_{t \in T} \lambda(t)$ is a finite routine which associates an element x_t of $\lambda(t)$ with each element t of T, such that $i_t(x_t) = i_{t'}(x_{t'})$ whenever $t, t' \in T$.

Motivation IV

- The various set-theoretic formalisations of BISH (Myhill, Friedman, Aczel, Feferman, Beeson, and Greenleaf) were influenced by ZF, and are "top-down" approaches to BISH.
- The type-theoretic interpretation of Bishop's set theory into the theory of setoids (Palmgren) is the standard way to understand Bishop sets. The identity type of MLTT expresses, in a proof-relevant way, the existence of the least reflexive relation on a type, a fact with no counterpart in Bishop's set theory. The free setoid on a type is definable, and the presentation axiom in setoids is provable.
- In MLTT the families of types over a type I is the type $I \to \mathcal{U}$, which is in $\mathcal{U}'$. In Bishop's set theory the set-character of the totality of all families of sets indexed by some set I is questionable from the predicative point of view.
- In MLTT no distinction between sets and classes, and between operations and functions.

Motivation V

- BST is an informal, constructive theory of totalities and assignment routines that serves as a completion of Bishop's set theory.
- Its aim is to fill in the "gaps" in Bishop's account of the set theory underlying BISH, and,
- to serve as an intermediate step between Bishop's informal set theory and a suitable i.e., an adequate and faithful, in the sense of Feferman, formalisation of BISH.

To assure faithfulness, we use concepts or principles that appear, explicitly or implicitly, in BISH.

Overview

- Fundamental notions of BST.
- Families of Bishop sets indexed by some set I and the family-maps between them. The corresponding $\sum$-and $\prod$-sets are introduced.
- Families of Bishop sets indexed by some directed set $(I, \preccurlyeq)$ and the family-maps between them. The corresponding $\sum^{\preccurlyeq}$-and $\prod^{\preccurlyeq}$-sets are introduced.
- Basic facts on Bishop spaces.
- Direct spectrum of Bishop spaces, and a canonical Bishop topology on the direct sum of Bishop spaces.
- Direct limit $\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i$ of a direct spectrum of Bishop spaces, universal property, the cofinality theorem for direct limits.
- Inverse limit $\underset{\leftarrow}{\mathrm{Lim}}\mathcal{F}_i$ of a contravariant direct spectrum of Bishop spaces, universal property.
- Duality principle between the inverse and direct limits of Bishop spaces.

Primitives of BST I

- (s, t).
- equality $:=$ between terms.
- $\mathtt{pr}_1(s,t) := s$ and $\mathtt{pr}_2(s,t) := t$.
- $\mathbb{N}$.
- Any other totality X is defined through a "membership-formula" $x \in X$.
- A defined equality on X is a formula $x =_X y$ that satisfies the properties of an equivalence relation.
- If X is a set and Y is a totality, an assignment routine $\alpha : X \rightsquigarrow Y$ from X to Y is a finite routine assigning an element y of Y, to each given element x of X. In this case we write $\alpha(x) := y$.
- If X, Y are sets, an assignment routine $f : X \rightsquigarrow Y$ is a function, if $f(x) =_Y f(x')$, for every $x, x' \in X$, such that $x =_X x'$. In this case we write $f : X \to Y$.

Primitives of BST II

- $\mathbb{F}(X, Y)$ with pointwise equality is a set (function extensionality).
- The (univalent) universe of sets $\mathbb{V}_0$ with equality

  $$X =_{\mathbb{V}_0} Y :\Leftrightarrow \exists_{f \in \mathbb{F}(X,Y)} \exists_{g \in \mathbb{F}(Y,X)} \big(g \circ f = \mathrm{id}_X \ \& \ f \circ g = \mathrm{id}_Y\big)$$

  is a class.
- If I is a set and $\mu_0 : I \rightsquigarrow \mathbb{V}_0$, a dependent assignment routine over μ_0 is an assignment routine μ_1 that assigns to each element i in I an element $\mu_1(i)$ in $\mu_0(i)$. We denote such a routine by

  $$\mu_1 : \bigcurlywedge_{i \in I} \mu_0(i),$$

  and their totality by $\mathbb{A}(I, \mu_0)$. If $\mu_1, \nu_1 : \bigcurlywedge_{i \in I} \mu_0(i)$, we define

  $$\mu_1 =_{\mathbb{A}(I,\mu_0)} \nu_1 :\Leftrightarrow \forall_{i \in I} \big(\mu_1(i) =_{\mu_0(i)} \nu_1(i)\big).$$

Primitives of BST III

- If X is a set, a subset of X is a pair (A, i_A), where A is a set and $i_A : A \hookrightarrow X$. If (A, i_A) and (B, i_B) are subsets of X, A is a subset of B, $A \subseteq B$, if there is $f : A \to B$ sttfdc

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, hook, "i_A"'] & & B \arrow[dl, hook', "i_B"] \\ & X. & \end{tikzcd}$$

  In this case we write $f : A \subseteq B$, and f is an embedding. Usually we write A instead of (A, i_A).
- The totality of the subsets of X is the powerset $\mathcal{P}(X)$, where $(A, i_A) =_{\mathcal{P}(X)} (B, i_B) :\Leftrightarrow A \subseteq B \ \& \ B \subseteq A$. If $f : A \subseteq B$ and $g : B \subseteq A$, we write $(f, g) : A =_{\mathcal{P}(X)} B$.

Primitives of BST IV

Since the membership condition of $\mathcal{P}(X)$ requires quantification over $\mathbb{V}_0$, the totality $\mathcal{P}(X)$ is a class.

$$\begin{tikzcd}
A \arrow[rr, hook, bend left, "f"] \arrow[dr, hook, "i_A"'] & & B \arrow[ll, hook, bend left, "g"] \arrow[dl, hook', "i_B"] \\
& X. &
\end{tikzcd}$$

$$(f,g) : A =_{\mathcal{P}(X)} B \Rightarrow (f,g) : A =_{\mathbb{V}_0} B.$$

If $(f', g') : A =_{\mathcal{P}(X)} B$, then $f = f'$ and $g = g'$.

Primitives of BST V

If $P(x)$ is an extensional property on X i.e.,

$$\forall_{x,y \in X}\big(x =_X y \Rightarrow (P(x) \Rightarrow P(y))\big),$$

the set X_P generated by P is defined by the membership-condition

$$x \in X_P :\Leftrightarrow x \in X \ \& \ P(x),$$

and the equality of X_P is inherited from the equality of X. Usually, we use the notation

$$\{x \in X \mid P(x)\} := X_P.$$

If X, Y are sets, their product $X \times Y$ is defined by

$$z \in X \times Y :\Leftrightarrow \exists_{x \in X}\exists_{y \in Y}\big(z := (x, y)\big),$$

and its equality is defined as usual.

If X is a set, then the property $P(x, y) :\Leftrightarrow x =_X y$ is an extensional property on $X \times X$ that generates the following subset of $X \times X$

$$D(X) := \big\{(x, y) \in X \times X \mid x =_X y\big\},$$

which we call the diagonal of X.

If I is a set, a family of sets indexed by I is a pair $\Lambda := (\lambda_0, \lambda_1)$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, and

$$\lambda_1 : \bigcurlywedge_{(i,j)\in D(I)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big),$$

such that, if $\lambda_1(i,j) := \lambda_{ij}$, for every $(i,j) \in D(I)$,

(a) For every $i \in I$, we have that $\lambda_{ii} := \mathrm{id}_{\lambda_0(i)}$.

(b) If $i =_I j$ and $j =_I k$, the following diagram commutes

$$\begin{tikzcd} \lambda_0(i) \arrow[d, "\lambda_{ij}"'] \arrow[dr, "\lambda_{ik}"] & \\ \lambda_0(j) \arrow[r, "\lambda_{jk}"'] & \lambda_0(k). \end{tikzcd}$$

If $i =_I j$, we call the function λ_{ij} the transport map from $\lambda_0(i)$ to $\lambda_0(j)$, and we call λ_1 the modulus of function-likeness of λ_0:

$$(\lambda_{ij}, \lambda_{ji}) : \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j).$$

An I-family of sets is called an I-set of sets, if

$$\forall_{i,j \in I}\big(\lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j) \Rightarrow i =_I j\big).$$

Functions on I are lifted to functions on $\lambda_0 I$, where

$$z \in \lambda_0 I :\Leftrightarrow \exists_{i \in I}\big(z := \lambda_0(i)\big)$$

$$\lambda_0(i) =_{\lambda_0 I} \lambda_0(j) :\Leftrightarrow \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j).$$

Let $\Lambda := (\lambda_0, \lambda_1)$ and $M := (\mu_0, \mu_1)$ be I-families of sets. A family-map from Λ to M is a d.a.r.

$$\Psi : \bigcurlywedge_{i \in I} \mathbb{F}\big(\lambda_0(i), \mu_0(i)\big)$$

such that for every $(i, j) \in D(I)$ tfdc

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\lambda_{ij}"] \arrow[d, "\Psi_i"'] & \lambda_0(j) \arrow[d, "\Psi_j"] \\
\mu_0(i) \arrow[r, "\mu_{ij}"'] & \mu_0(j).
\end{tikzcd}$$

$\mathtt{Map}_I(\Lambda, M)$ with $\Psi =_{\mathtt{Map}_I(\Lambda,M)} \Xi :\Leftrightarrow \forall_{i \in I}\big(\Psi_i =_{\mathbb{F}(\lambda_0(i),\mu_0(i))} \Xi_i\big)$.

$\Psi : \Lambda \Rightarrow M$ denotes an element of $\mathtt{Map}_I(\Lambda, M)$.

$$
\begin{tikzcd}
\lambda_0(i) \arrow[r, "\lambda_{ij}"] \arrow[d, "\Psi_i"] \arrow[dd, bend right, "(\Xi \circ \Psi)_i"'] & \lambda_0(j) \arrow[d, "\Psi_j"'] \arrow[dd, bend left, "(\Xi \circ \Psi)_j"] \\
\mu_0(i) \arrow[r, "\mu_{ij}"'] \arrow[d, "\Xi_i"] & \mu_0(j) \arrow[d, "\Xi_j"'] \\
\nu_0(i) \arrow[r, "\nu_{ij}"'] & \nu_0(j),
\end{tikzcd}
$$

$$\mathrm{Id}_\Lambda : \bigcurlywedge_{i \in I} \mathbb{F}\big(\lambda_0(i), \lambda_0(i)\big), \quad \mathrm{Id}_\Lambda(i) := \mathrm{id}_{\lambda_0(i)}, \quad i \in I.$$

$\mathtt{Fam}(I)$ the totality of I-families of sets with equality

$\Lambda =_{\mathtt{Fam}(I)} M :\Leftrightarrow \exists_{\Phi \in \mathtt{Map}_I(\Lambda, M)} \exists_{\Xi \in \mathtt{Map}_I(M, \Lambda)} \big(\Phi \circ \Xi = \mathrm{id}_M \ \& \ \Xi \circ \Phi = \mathrm{id}_\Lambda\big).$

If $\mathtt{Fam}(I)$ was a set, the constant I-family with value $\mathtt{Fam}(I)$ would be defined though a totality in which it belongs to.

Let $\Lambda := (\lambda_0, \lambda_1)$ be an I-family of sets.

The exterior union $\sum_{i \in I} \lambda_0(i)$ of Λ is defined by

$$w \in \sum_{i \in I} \lambda_0(i) :\Leftrightarrow \exists_{i \in I} \exists_{x \in \lambda_0(i)} \big(w := (i, x)\big),$$

$$(i, x) =_{\sum_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow i =_I j \ \& \ \lambda_{ij}(x) =_{\lambda_0(j)} y.$$

The totality $\prod_{i \in I} \lambda_0(i)$ of dependent functions over Λ is defined by

$$\Phi \in \prod_{i \in I} \lambda_0(i) :\Leftrightarrow \Phi \in \mathbb{A}(I, \lambda_0) \ \& \ \forall_{(i,j) \in D(I)} \big(\Phi_j =_{\lambda_0(j)} \lambda_{ij}(\Phi_i)\big),$$

and it is equipped with the equality of $\mathbb{A}(I, \lambda_0)$.

Proposition

Let $\Lambda := (\lambda_0, \lambda_1)$, $M := (\mu_0, \mu_1) \in \mathtt{Fam}(I)$, *and* $\Psi \in \mathtt{Map}_I(\Lambda, M)$.

(i) *For every* $i \in I$ *the a.r.* $e_i^\Lambda : \lambda_0(i) \rightsquigarrow \sum_{i \in I} \lambda_0(i)$, *defined by* $x \mapsto (i, x)$, *is an embedding of* $\lambda_0(i)$ *into* $\sum_{i \in I} \lambda_0(i)$.

(ii) *The a.r.* $\Sigma\Psi : \sum_{i \in I} \lambda_0(i) \rightsquigarrow \sum_{i \in I} \mu_0(i)$,

$$\Sigma\Psi(i, x) := (i, \Psi_i(x)),$$

is a function from $\sum_{i \in I} \lambda_0(i)$ *to* $\sum_{i \in I} \mu_0(i)$, *s.t. for every* $i \in I$ *tfdc*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] \arrow[d, "e_i^\Lambda"'] & \mu_0(i) \arrow[d, "e_i^M"] \\
\sum_{i \in I} \lambda_0(i) \arrow[r, "\Sigma\Psi"'] & \sum_{i \in I} \mu_0(i).
\end{tikzcd}$$

(iii) *If every* Ψ_i *is an embedding, then* $\Sigma\psi$ *is an embedding.*

Proposition

Let $\Lambda := (\lambda_0, \lambda_1)$, $M := (\mu_0, \mu_1) \in \mathtt{Fam}(I)$, *and* $\Psi \in \mathtt{Map}_I(\Lambda, M)$.

(i) For every $i \in I$ *the a.r.* $\pi_i^{\Lambda} : \prod_{i \in I} \lambda_0(i) \rightsquigarrow \lambda_0(i)$, *defined by* $\Theta \mapsto \Theta_i$, *is a function from* $\prod_{i \in I} \lambda_0(i)$ *to* $\lambda_0(i)$.

(ii) The a.r. $\Pi\Psi : \prod_{i \in I} \lambda_0(i) \rightsquigarrow \prod_{i \in I} \mu_0(i)$,

$$[\Pi\Psi(\Theta)]_i := \Psi_i(\Theta_i),$$

is a function from $\prod_{i \in I} \lambda_0(i)$ *to* $\prod_{i \in I} \mu_0(i)$, *s.t. for every* $i \in I$ *tfdc*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] & \mu_0(i) \\
\prod_{i \in I} \lambda_0(i) \arrow[u, "\pi_i^{\Lambda}"] \arrow[r, "\Pi\Psi"'] & \prod_{i \in I} \mu_0(i). \arrow[u, "\pi_i^{M}"']
\end{tikzcd}$$

(iii) If every Ψ_i *is an embedding, then* $\Pi\Psi$ *is an embedding.*

- Distributivity of $\prod$ over $\sum$ in [10].
- Yoneda lemma in [12].
- Spectra of Bishop spaces, canonical Bishop topology on the corresponding sums and products of Bishop spaces.

- A direct family of sets is a variation of the notion of a set-indexed family of sets.
- A family of sets over a partial order is also used in the definition of a certain Kripke model for intuitionistic predicate logic, and the corresponding transport maps $\lambda_{ij}^{\preccurlyeq}$ are called transition functions (see TvD88, p. 85).
- Let $(I, \preccurlyeq_I)$ be a directed set i.e., $i \preccurlyeq_I j$ is a binary relation on I which is extensional i.e.,

$$\forall_{i,j \in I}\big(i =_I i' \ \& \ j =_I j' \ \& \ i \preccurlyeq j \Rightarrow i' \preccurlyeq j'\big),$$

which is also reflexive, transitive, and

$$\forall_{i,j \in I}\exists_{k \in I}\big(i \preccurlyeq_I k \ \& \ j \preccurlyeq_I k\big).$$

Since $i \preccurlyeq_I j$ is extensional, it generates the following extensional subset of $I \times I$

$$\preccurlyeq(I) := \big\{(i,j) \in I \times I \mid i \preccurlyeq_I j\big\}.$$

A direct family of sets indexed by $(I, \preccurlyeq_I)$ is a pair $\Lambda^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq})$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, and

$$\lambda_1^{\preccurlyeq} : \bigcurlywedge_{(i,j) \in \preccurlyeq(I)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big), \quad \lambda_1^{\preccurlyeq}(i,j) := \lambda_{ij}^{\preccurlyeq}, \quad (i,j) \in \preccurlyeq(I),$$

such that the following conditions hold:

(a) For every $i \in I$, we have that $\lambda_{ii}^{\preccurlyeq} := \mathrm{id}_{\lambda_0(i)}$.

(b) If $i \preccurlyeq_I j$ and $j \preccurlyeq_I k$, the following diagram commutes

$$\begin{tikzcd}
\lambda_0(i) \arrow[d, "\lambda_{ij}^{\preccurlyeq}"'] \arrow[dr, "\lambda_{ik}^{\preccurlyeq}"] & \\
\lambda_0(j) \arrow[r, "\lambda_{jk}^{\preccurlyeq}"'] & \lambda_0(k).
\end{tikzcd}$$

$\Psi : \Lambda^{\preccurlyeq} \Rightarrow M^{\preccurlyeq}$ a direct family-map

$\mathtt{Map}_{(I,\preccurlyeq_I)}(\Lambda^{\preccurlyeq}, M^{\preccurlyeq})$

$\mathtt{Fam}(I, \preccurlyeq_I)$

The direct sum $\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)$ over $\Lambda^{\preccurlyeq}$ is the totality $\sum_{i\in I} \lambda_0(i)$ equipped with the equality

$$(i, x) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (j, y) :\Leftrightarrow \exists_{k\in I}\big(i \preccurlyeq k \;\&\; j \preccurlyeq k \;\&\; \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)\big).$$

The totality $\prod_{i\in I}^{\preccurlyeq} \lambda_0(i)$ of dependent functions over $\Lambda^{\preccurlyeq}$ is defined by

$$\Phi \in \prod_{i\in I}^{\preccurlyeq} \lambda_0(i) :\Leftrightarrow \Phi \in \mathbb{A}(I, \lambda_0) \;\&\; \forall_{(i,j)\in\preccurlyeq(I)}\big(\Phi_j =_{\lambda_0(j)} \lambda_{ij}^{\preccurlyeq}(\Phi_i)\big),$$

and it is equipped with the equality of $\mathbb{A}(I, \lambda_0)$. The totality $\prod_{i\in I}^{\succcurlyeq} \lambda_0(i)$ is defined similarly.

Similar propositions for $e_i^{\Lambda^{\preccurlyeq}}, \Sigma^{\preccurlyeq}\Psi, \pi_i^{\Lambda^{\preccurlyeq}}, \Pi^{\preccurlyeq}\Psi$.

A Bishop space is a pair $\mathcal{F} := (X, F)$, where $F \subseteq \mathbb{F}(X)$, which is called a Bishop topology that satisfies the following conditions:

(BS_1) If $a \in \mathbb{R}$, then $\overline{a}^X \in F$.

(BS_2) If $f, g \in F$, then $f + g \in F$.

(BS_3) If $f \in F$ and $\phi \in \mathrm{Bic}(\mathbb{R})$, then $\phi \circ f \in F$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "F \ni \phi \circ f"'] & \mathbb{R} \arrow[d, "\phi \in \mathrm{Bic}(\mathbb{R})"] \\ & \mathbb{R}. \end{tikzcd}$$

(BS_4) $\overline{F} = F$.

The least topology $\bigvee F_0$ generated by some $F_0 \subseteq \mathbb{F}(X)$, is defined by turning the above clauses to inductive rules plus

$$\frac{f_0 \in F_0}{f_0 \in \bigvee F_0}.$$

Corresponding induction principle.

If $\mathcal{F} := (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, a function $h : X \to Y$ is called a Bishop morphism, if $\forall_{g \in G}(g \circ h \in F)$

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[dr, swap, "F \ni g \circ h"] & Y \arrow[d, "g \in G"] \\
 & \mathbb{R}.
\end{tikzcd}$$

We denote by $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$ the set of Bishop morphisms from $\mathcal{F}$ to $\mathcal{G}$. If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, the induced mapping $h^* : G \to F$ from h is defined, for every $g \in G$, by $h^*(g) := g \circ h$. $\bigvee$-lifting of morphisms

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[dr, swap, "F \ni g_0 \circ h"] & Y \arrow[d, "g_0 \in G_0"] \\
 & \mathbb{R}.
\end{tikzcd}$$

$h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ is a Bishop isomorphism if and only if it is open i.e.,

$$\forall_{f \in F} \exists_{g \in G} \big(f = g \circ h\big).$$

According to the $\bigvee$-lifting of openness, if $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ is a surjection, and if $F = \bigvee F_0$ it suffices to prove the openness property only for the subbase F_0 of F i.e.,

$$\forall_{f_0 \in F_0} \exists_{g \in G} \big(f_0 = g \circ h\big) \Rightarrow \forall_{f \in \bigvee F_0} \exists_{g \in G} \big(f = g \circ h\big),$$

$$F \times G := \bigvee [\{f \circ \pi_1, \mid f \in F\} \cup \{g \circ \pi_2 \mid g \in G\}] =: \bigvee_{f \in F}^{g \in G} f \circ \pi_1, g \circ \pi_2$$

$$F_{|A} = \bigvee \{f_{|A} \mid f \in F\} =: \bigvee_{f \in F} f_{|A},$$

$$F \to G := \bigvee \left\{\phi_{x,g} \mid x \in X, g \in G\right\} := \bigvee_{x \in X}^{g \in G} \phi_{x,g},$$

where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the projections on X and Y, respectively, $f_{|A} := f \circ i_A$

$$\begin{tikzcd} A \arrow[r, hook, "i_A"] \arrow[rr, bend right, "f_{|A}"'] & X \arrow[r, "f"] & \mathbb{R}, \end{tikzcd}$$

and $\phi_{x,g} : \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \to \mathbb{R}$ is defined by

$$\phi_{x,g}(h) = g(h(x)).$$

$$\bigvee F_0 \times \bigvee G_0 := \bigvee [\{f_0 \circ \pi_1, \mid f_0 \in F_0\} \cup \{g_0 \circ \pi_2 \mid g_0 \in G_0\}]$$

$$=: \bigvee_{f_0 \in F_0}^{g_0 \in G_0} f_0 \circ \pi_1, g_0 \circ \pi_2,$$

$$\left(\bigvee F_0\right)_{|A} = \bigvee \{f_{0|A} \mid f_0 \in F_0\} =: \bigvee_{f_0 \in F_0} f_{0|A},$$

$$F \to \bigvee G_0 = \bigvee \left\{\phi_{x,g_0} \mid x \in X, g_0 \in G_0\right\} := \bigvee_{x \in X}^{g_0 \in G_0} \phi_{x,g_0}.$$

Let $(I, \preccurlyeq)$ be a directed set, and let $\Lambda^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq})$, $M^{\preccurlyeq} := (\mu_0, \mu_1^{\preccurlyeq})$ be $(I, \preccurlyeq)$-families of sets.

A family of Bishop topologies associated to $\Lambda^{\preccurlyeq}$ is a pair $\Phi^{\Lambda^{\preccurlyeq}} := \big(\phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}}\big)$, where $\phi_0^{\Lambda^{\preccurlyeq}} : I \rightsquigarrow \mathbb{V}_0$ and

$$\phi_1^{\Lambda^{\preccurlyeq}} : \bigcurlywedge_{(i,j) \in \preccurlyeq(I)} \mathbb{F}\big(\phi_0^{\Lambda^{\preccurlyeq}}(j), \phi_0^{\Lambda^{\preccurlyeq}}(i)\big)$$

such that the following conditions hold:

(i) $\phi_0^{\Lambda^{\preccurlyeq}}(i) := F_i \subseteq \mathbb{F}(\lambda_0(i), \mathbb{R})$, and $\mathcal{F}_i := (\lambda_0(i), F_i)$ is a Bishop space, for every $i \in I$.
(ii) $\lambda_{ij}^{\preccurlyeq} \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{F}_j)$, for every $(i,j) \in \preccurlyeq(I)$.
(iii) $\phi_1^{\Lambda^{\preccurlyeq}}(i,j) := \big(\lambda_{ij}^{\preccurlyeq}\big)^*$, for every $(i,j) \in \preccurlyeq(I)$, where, if $f \in F_j$, $\big(\lambda_{ij}^{\preccurlyeq}\big)^*(f) := f \circ \lambda_{ij}^{\preccurlyeq}$.

The structure $S^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ is called a direct spectrum over $(I, \preccurlyeq)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j) \in \preccurlyeq(I)}$.

- If $T^{\preccurlyeq} := (\mu_0, \mu_1, \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}})$ is an $(I, \preccurlyeq)$-spectrum with Bishop spaces $(\mathcal{G}_i)_{i \in I}$ and Bishop morphisms $(\mu_{ij}^{\preccurlyeq})_{(i,j) \in \preccurlyeq(I)}$, a direct spectrum-map Ψ from $S^{\preccurlyeq}$ to $T^{\preccurlyeq}$ is a direct family-map $\Psi : \Lambda^{\preccurlyeq} \Rightarrow M^{\preccurlyeq}$.
- A direct spectrum-map $\Psi : S^{\preccurlyeq} \Rightarrow T^{\prec}$ is called continuous, if $\forall_{i \in I}\big(\Psi_i \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{G}_i)\big)$.
- A contravariant direct spectrum $S^{\succcurlyeq} := (\lambda_0, \lambda_1^{\succcurlyeq}; \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ over $(I, \preccurlyeq)$, and a contravariant direct spectrum-map $\Psi : \Lambda^{\succcurlyeq} \Rightarrow M^{\succcurlyeq}$ are defined similarly.

Remark

Let $(I, \preccurlyeq)$ be a directed set and $S^{\preccurlyeq} := (\lambda_0, \lambda_1; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ an $(I, \preccurlyeq)$-spectrum with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j) \in \preccurlyeq(I)}$. If $\Theta \in \prod_{i \in I}^{\succcurlyeq} F_i$, the assignment routine $f_\Theta : \left(\sum_{i \in I}^{\preccurlyeq} \lambda_0(i)\right) \rightsquigarrow \mathbb{R}$, defined by

$$f_\Theta(i, x) := \Theta_i(x),$$

is a function from $\sum_{i \in I}^{\preccurlyeq} \lambda_0(i)$ to $\mathbb{R}$.

Proof.

Let $(i, x) =_{\sum_{i \in I}^{\preccurlyeq} \lambda_0(i)} (j, y) \Leftrightarrow \exists_{k \succcurlyeq i,j} \big(\lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{jk}^{\preccurlyeq}(y)\big)$. Since

$$\Theta_i = \phi_{ki}^{\succcurlyeq}(\Theta_k) := (\lambda_{ik}^{\preccurlyeq})^*(\Theta_k) := \Theta_k \circ \lambda_{ik}^{\preccurlyeq},$$

and similarly $\Theta_j = \Theta_k \circ \lambda_{jk}^{\preccurlyeq}$, we have that

$$\Theta_i(x) = \left[\Theta_k \circ \lambda_{ik}^{\preccurlyeq}\right](x) := \Theta_k\big(\lambda_{ik}^{\preccurlyeq}(x)\big) = \Theta_k\big(\lambda_{jk}^{\preccurlyeq}(y)\big)$$

$$:= \left[\Theta_k \circ \lambda_{jk}^{\preccurlyeq}\right](y) = \Theta_j(y).$$

Let $(I, \preccurlyeq)$ be a directed set and $S^{\preccurlyeq} := (\lambda_0, \lambda_1; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ an $(I, \preccurlyeq)$-spectrum with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j) \in \preccurlyeq(I)}$. The Bishop space

$$\sum_{i \in I}^{\preccurlyeq} \mathcal{F}_i := \bigg(\sum_{i \in I}^{\preccurlyeq} \lambda_0(i), \sum_{i \in I}^{\preccurlyeq} F_i \bigg),$$

$$\sum_{i \in I}^{\preccurlyeq} F_i := \bigvee_{\Theta \in \prod_{i \in I}^{\succcurlyeq} F_i} f_{\Theta},$$

is called the sum Bishop space of $S^{\preccurlyeq}$.

If $S^{\succcurlyeq}$ is a contravariant direct spectrum over $(I, \preccurlyeq)$, the sum Bishop space of $S^{\succcurlyeq}$ is defined dually.

Proposition

Let $S^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ *and* $T^{\preccurlyeq} := (\mu_0, \mu_1^{\preccurlyeq}; \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}})$ *be direct spectra over* $(I, \preccurlyeq)$*, and let* $\Psi : S^{\preccurlyeq} \Rightarrow T^{\preccurlyeq}$.

(i) *If* $i \in I$*, then* $e_i^{\Lambda^{\preccurlyeq}} \in \mathrm{Mor}\big(\mathcal{F}_i, \sum_{i\in I}^{\preccurlyeq} \mathcal{F}_i\big)$.

(ii) *If* Ψ *is continuous, then* $\Sigma^{\preccurlyeq}\Psi \in \mathrm{Mor}\big(\sum_{i\in I}^{\preccurlyeq} \mathcal{F}_i, \sum_{i\in I}^{\preccurlyeq} \mathcal{G}_i\big)$.

Proof.

(i) $\forall_{\Theta\in\prod_{i\in I}^{\succcurlyeq} F_i}\big(f_\Theta \circ e_i^{\Lambda^{\preccurlyeq}} \in F_i\big)$. If $x \in \lambda_0(i)$, then
$\big(f_\Theta \circ e_i^{\Lambda^{\preccurlyeq}}\big)(x) := f_\Theta(i, x) := \Theta_i(x)$, hence $f_\Theta \circ e_i^{\Lambda^{\preccurlyeq}} := \Theta_i \in F_i$.
(ii)

$$\forall_{H\in\prod_{i\in I}^{\succcurlyeq} G_i}\left(g_H \circ \Sigma^{\preccurlyeq}\Psi \in \sum_{i\in I}^{\preccurlyeq} F_i\right).$$

If $i \in I$ and $x \in \lambda_0(i)$, we have that $H^* \in \prod_{i\in I}^{\succcurlyeq} F_i$, and
$g_H \circ \Sigma^{\preccurlyeq}\Psi := f_{H^*} \in \sum_{i\in I}^{\preccurlyeq} F_i$

$$\big(g_H \circ \Sigma^{\preccurlyeq}\Psi\big)(i, x) := g_H(i, \Psi_i(x)) := H_i(\Psi_i(x)) := (H_i \circ \Psi_i)(x) := f_{H^*}(i, x)$$

Let X be a set and $\omega_X : X \rightsquigarrow \mathbb{V}_0$ the assignment routine defined by

$$\omega_X(x) := \{x' \in X \mid x' =_X x\},$$

for every $x \in X$. If X is clear from the context, we write ω instead of ω_X. The totality ωX is defined by

$$z \in \omega X :\Leftrightarrow \exists_{x \in X}\big(z := \omega_X(x)\big),$$

$$\omega_X(x) =_{\omega X} \omega_X(x) :\Leftrightarrow \omega_X(x) =_{\mathcal{P}(X)} \omega_X(x').$$

Clearly, $\omega_X(x) =_{\mathcal{P}(X)} \omega_X(x') \Leftrightarrow x =_X x'$, hence ω is an X-set of subsets of X.

Let $(I, \preccurlyeq)$ be a directed set and $S^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ a direct spectrum over $(I, \preccurlyeq)$. If $\omega : \sum_{i\in I}^{\preccurlyeq} \lambda_0(i) \rightsquigarrow \mathcal{P}(\sum_{i\in I}^{\preccurlyeq})$ is defined by

$$\omega(i,x) := \left\{(j,y) \in \sum_{i\in I}^{\preccurlyeq} \lambda_0(i) \mid (j,y) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (i,x)\right\},$$

for every $(i,x) \in \sum_{i\in I}^{\preccurlyeq} \lambda_0(i)$, the direct limit set $\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)$ of $S^{\preccurlyeq}$ is defined by

$$\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i) := \omega \sum_{i\in I}^{\preccurlyeq} \lambda_0(i).$$

$$\omega(i,x) =_{\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)} \omega(j,y) :\Leftrightarrow \omega(i,x) =_{\mathcal{P}\left(\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)\right)} \omega(j,y)$$

$$\Leftrightarrow (i,x) =_{\sum_{i\in I}^{\preccurlyeq} \lambda_0(i)} (j,y).$$

Remark

Let $(I, \preccurlyeq)$ be a directed set and $S^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ a direct spectrum over $(I, \preccurlyeq)$. For every $i \in I$, the assignment routine $\omega_i : \lambda_0(i) \rightsquigarrow \underset{\rightarrow}{\mathrm{Lim}} \lambda_0(i)$, defined by $\omega_i(x) := \omega(i, x)$, for every $x \in \lambda_0(i)$, is a function from $\lambda_0(i)$ to $\underset{\rightarrow}{\mathrm{Lim}} \lambda_0(i)$.

Proof.

If $x, x' \in \lambda_0(i)$ such that $x =_{\lambda_0(i)} x'$, then

$$\begin{aligned}
\omega_i(x) =_{\underset{\rightarrow}{\mathrm{Lim}} \lambda_0(i)} \omega_i(x') &:\Leftrightarrow \omega(i, x) =_{\underset{\rightarrow}{\mathrm{Lim}} \lambda_0(i)} \omega(i, x') \\
&\Leftrightarrow (i, x) =_{\sum_{i \in I}^{\preccurlyeq} \lambda_0(i)} (i, x') \\
&:\Leftrightarrow \exists_{k \in I} \big(i \preccurlyeq k \ \&\ \lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{ik}^{\preccurlyeq}(x')\big),
\end{aligned}$$

which holds, since $\lambda_{ik}^{\preccurlyeq}$ is a function, and hence if $x =_{\lambda_0(i)} x'$, then $\lambda_{ik}^{\preccurlyeq}(x) =_{\lambda_0(k)} \lambda_{ik}^{\preccurlyeq}(x')$. $\square$

Let $(I, \preccurlyeq)$ be a directed set and $S^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ a directed spectrum over $(I, \preccurlyeq)$. The direct limit Bishop space of $S^{\preccurlyeq}$ is the Bishop space

$$\underset{\rightarrow}{\mathrm{Lim}} \mathcal{F}_i := \big(\underset{\rightarrow}{\mathrm{Lim}} \lambda_0(i), \underset{\rightarrow}{\mathrm{Lim}} F_i\big),$$

$$\underset{\rightarrow}{\mathrm{Lim}} F_i := \bigvee_{\Theta \in \prod_{i \in I}^{\succcurlyeq} F_i} \omega f_{\Theta},$$

$$\big(\omega f_{\Theta}\big)\omega(i, x) := f_{\Theta}(i, x) := \Theta_i(x).$$

Proposition (Universal property of the direct limit)

If $S^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ is a direct spectrum over $(I, \preccurlyeq)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j) \in \preccurlyeq(I)}$, its direct limit $\underset{\rightarrow}{\mathrm{Lim}} \mathcal{F}_i$ satisfies the universal property of direct limits i.e.,

(i) *For every $i \in I$, we have that $\omega_i \in \mathrm{Mor}(\mathcal{F}_i, \underset{\rightarrow}{\mathrm{Lim}} \mathcal{F}_i)$.*

(ii) *If $i \preccurlyeq j$, the following left diagram commutes*

$$
\begin{tikzcd}
& \underset{\rightarrow}{\mathrm{Lim}} \lambda_0(i) & \\
\lambda_0(i) \arrow[ur, "\omega_i"] \arrow[rr, "\lambda_{ij}^{\preccurlyeq}"'] & & \lambda_0(j) \arrow[ul, "\omega_j"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& Y & \\
\lambda_0(i) \arrow[ur, "\varepsilon_i"] \arrow[rr, "\lambda_{ij}^{\preccurlyeq}"'] & & \lambda_0(j). \arrow[ul, "\varepsilon_j"']
\end{tikzcd}
$$

(iii) If $\mathcal{G} := (Y, G)$ is a Bishop space and $\varepsilon_i : \lambda_0(i) \to Y \in \mathrm{Mor}(\mathcal{F}_i, \mathcal{G})$, for every $i \in I$, such that if $i \preccurlyeq j$, the previous right diagram commutes, there is a unique function $h : \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i) \to Y \in \mathrm{Mor}(\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i, \mathcal{G})$ that makes the following diagrams commutative

$$\begin{tikzcd}
 & Y & \\
\lambda_0(i) \arrow[ur, "\varepsilon_i"] \arrow[rr, "\lambda_{ij}^{\preccurlyeq}"'] \arrow[dr, "\omega_i"'] & & \lambda_0(j), \arrow[ul, "\varepsilon_j"'] \arrow[dl, "\omega_j"] \\
 & \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i) \arrow[uu, dashed, "h"'] &
\end{tikzcd}$$

Theorem

Let $S^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ *and* $T^{\preccurlyeq} := (\mu_0, \mu_1^{\preccurlyeq}; \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}})$ *be direct spectra over* $(I, \preccurlyeq)$, *and let* $\Psi : S^{\preccurlyeq} \Rightarrow T^{\preccurlyeq}$.

(i) *There is a unique function* $\Psi_{\rightarrow} : \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i) \rightarrow \underset{\rightarrow}{\mathrm{Lim}}\mu_0(i)$ *such that, for every* $i \in I$, *the following diagram commutes*

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] \arrow[d, "\omega_{S^{\preccurlyeq},i}"'] & \mu_0(i) \arrow[d, "\omega_{T^{\preccurlyeq},i}"] \\
\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i) \arrow[r, dashed, "\Psi_{\rightarrow}"'] & \underset{\rightarrow}{\mathrm{Lim}}\mu_0(i).
\end{tikzcd}$$

(ii) *If* Ψ *is continuous, then* $\Psi_{\rightarrow} \in \mathrm{Mor}(\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i, \underset{\rightarrow}{\mathrm{Lim}}\mathcal{G}_i)$.

(iii) *If* Ψ_i *is an injection, for every* $i \in I$, *then* $\Psi_{\rightarrow}$ *is an injection.*

Proposition

If $S^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$, $T^{\preccurlyeq} := (\mu_0, \mu_1^{\preccurlyeq}; \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}})$ *and* $U^{\preccurlyeq} := (\nu_0, \nu_1^{\preccurlyeq}; \phi_0^{N^{\preccurlyeq}}, \phi_1^{N^{\preccurlyeq}})$ *are direct spectra over* $(I, \preccurlyeq)$, *and if* $\Psi : S^{\preccurlyeq} \Rightarrow T^{\preccurlyeq}$ *and* $\Xi : T^{\preccurlyeq} \Rightarrow U^{\preccurlyeq}$, *then we have that*

$$(\Xi \circ \Psi)_{\rightarrow} := \Xi_{\rightarrow} \circ \Psi_{\rightarrow}$$

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] \arrow[rr, bend left, "(\Xi \circ \Psi)_i"] \arrow[d, "\omega_{S^{\preccurlyeq},i}"'] & \mu_0(i) \arrow[r, "\Xi_i"] \arrow[d, "\omega_{T^{\preccurlyeq},i}"'] & \nu_0(i) \arrow[d, "\omega_{U^{\preccurlyeq},i}"] \\
\underset{\rightarrow}{\mathrm{Lim}} \lambda_0(i) \arrow[r, dashed, "\Psi_{\rightarrow}"'] \arrow[rr, dashed, bend right, "(\Xi \circ \Psi)_{\rightarrow}"'] & \underset{\rightarrow}{\mathrm{Lim}} \mu_0(i) \arrow[r, dashed, "\Xi_{\rightarrow}"'] & \underset{\rightarrow}{\mathrm{Lim}} \nu_0(i).
\end{tikzcd}$$

Let $(I, \preccurlyeq)$ be a directed set and $(J, e) \subseteq I$, and let $j \preccurlyeq j' :\Leftrightarrow e(j) \preccurlyeq e(j')$, for every $j, j' \in J$. We say that J is cofinal in I, if there is a function $\mathtt{cof}_J : I \to J$, which we call a modulus of cofinality of J in I, that satisfies the following conditions:

(i) $\forall_{j \in J}\big(\mathtt{cof}_J(e(j)) =_J j\big)$

$$\begin{tikzcd} J \arrow[d, hook, "e"'] \arrow[dr, "\mathrm{id}_J"] & \\ I \arrow[r, "\mathtt{cof}_J"'] & J. \end{tikzcd}$$

(ii) $\forall_{i, i' \in I}\big(i \preccurlyeq i' \Rightarrow \mathtt{cof}_J(i) \preccurlyeq \mathtt{cof}_J(i')\big)$.

(iii) $\forall_{i \in I}\big(i \preccurlyeq e(\mathtt{cof}_J(i))\big)$.

We denote the fact that J is cofinal in I by $(J, e, \mathtt{cof}_J) \subseteq^{\mathtt{cof}} I$, or, simpler, by $J \subseteq^{\mathtt{cof}} I$.

If $(I, \preccurlyeq)$ is directed and $(J, e, \mathtt{cof}_J) \subseteq^{\mathtt{cof}} I$, $(J, \preccurlyeq)$ is directed.

Proposition

Let $S^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ be a direct spectrum over $(I, \preccurlyeq)$, and $(J, e, \mathtt{cof}_J) \subseteq^{\mathtt{cof}} I$. The relative spectrum of $S^{\preccurlyeq}$ to J is

$$S_{|J}^{\preccurlyeq} := \left((\lambda_0)_{|J}, (\lambda_1)_{|J}^{\preccurlyeq}; \phi_0^{\Lambda_{|J}^{\preccurlyeq}}, \phi_1^{\Lambda_{|J}^{\preccurlyeq}}\right),$$

where $(\lambda_0)_{|J} : J \rightsquigarrow \mathbb{V}_0$ is defined by $(\lambda_0)_{|J}(j) := \lambda_0(e(j))$, and

$$(\lambda_1)_{|J}^{\preccurlyeq} : \bigcurlywedge_{(j,j') \in \preccurlyeq(J)} \mathbb{F}\big(\lambda_0(e(j)), \lambda_0(e(j'))\big), \quad (\lambda_1)_{|J}^{\preccurlyeq}(j, j') := \lambda_{jj'}^{\preccurlyeq} := \lambda_{e(j)e(j')}^{\preccurlyeq}$$

and $\phi_0^{\Lambda_{|J}^{\preccurlyeq}} : J \rightsquigarrow \mathbb{V}_0$ is defined by $\phi_0^{\Lambda_{|J}^{\preccurlyeq}}(j) := F_j := F_{e(j)} := \phi_0^{\Lambda^{\preccurlyeq}}(e(j))$

$$\phi_1^{\Lambda_{|J}^{\preccurlyeq}} : \bigcurlywedge_{(j,j') \in \preccurlyeq(J)} \mathbb{F}\big(F_{e(j')}, F_{e(j)}\big),$$

$$\phi_1^{\Lambda_{|J}^{\preccurlyeq}}(j, j') := \phi_1^{\Lambda^{\preccurlyeq}}(e(j), e(j')) := (\lambda_{e(j)e(j')}^{\preccurlyeq})^*$$

Theorem

Let $S^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ *be a direct spectrum over* $(I, \preccurlyeq)$*,* $(J, e, \mathtt{cof}_J)$ *cofinal in* I*, and* $S_{|J}^{\preccurlyeq} := \big((\lambda_0)_{|J}, (\lambda_1)_{|J}^{\preccurlyeq}; \phi_0^{\Lambda_{|J}^{\preccurlyeq}}, \phi_1^{\Lambda_{|J}^{\preccurlyeq}}\big)$ *the relative spectrum of* $S^{\preccurlyeq}$ *to* J*. Then*

$$\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_j \simeq \underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i.$$

Proof.

We define the assignment routine $\phi : \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(j) \rightsquigarrow \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)$ by

$$\phi\big(\omega_{S_{|J}^{\preccurlyeq}}(j, y)\big) := \omega_{S^{\preccurlyeq}}(e(j), y),$$

$$\begin{tikzcd}
& \lambda_0(j) \arrow[dl, "\omega_{S_{|J}^{\preccurlyeq}, j}"'] \arrow[dr, "\omega_{S^{\preccurlyeq}, j}"] & \\
\underset{\rightarrow}{\mathrm{Lim}}\lambda_0(j) \arrow[rr, "\phi"'] & & \underset{\rightarrow}{\mathrm{Lim}}\lambda_0(i)
\end{tikzcd}$$

If $S^{\succcurlyeq} := (\lambda_0, \lambda_1^{\succcurlyeq}; \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ is a contravariant $(I, \preccurlyeq)$-spectrum with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ji}^{\succcurlyeq})_{(i,j) \in \preccurlyeq(I)}$, the inverse limit of $S^{\succcurlyeq}$ is the Bishop space

$$\underset{\leftarrow}{\mathrm{Lim}} \mathcal{F}_i := \big(\underset{\leftarrow}{\mathrm{Lim}} \lambda_0(i),\ \underset{\leftarrow}{\mathrm{Lim}} F_i\big),$$

$$\underset{\leftarrow}{\mathrm{Lim}} \lambda_0(i) := \prod_{i \in I}^{\succcurlyeq} \lambda_0(i) \quad \& \quad \underset{\leftarrow}{\mathrm{Lim}} F_i := \bigvee_{i \in I}^{f \in F_i} f \circ \pi_i^{\Lambda^{\succcurlyeq}}.$$

We write π_i instead of $\pi_i^{\Lambda^{\succcurlyeq}}$ for the function $\pi_i^{\Lambda^{\succcurlyeq}} : \prod_{i \in I}^{\succcurlyeq} \lambda_0(i) \to \lambda_0(i)$, which is defined by the rule $\Phi \mapsto \Phi_i$.

Proposition (Universal property of the inverse limit)

If $S^{\succcurlyeq} := (\lambda_0, \lambda_1^{\succcurlyeq}; \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ is a contravariant direct spectrum over $(I, \preccurlyeq)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ji}^{\succcurlyeq})_{(i,j) \in \preccurlyeq(I)}$, its inverse limit $\underleftarrow{\mathrm{Lim}} \mathcal{F}_i$ satisfies the universal property of inverse limits i.e.,

(i) For every $i \in I$, we have that $\pi_i \in \mathrm{Mor}(\underleftarrow{\mathrm{Lim}} \mathcal{F}_i, \mathcal{F}_i)$.

(ii) If $i \preccurlyeq j$, the following left diagram commutes

$$\begin{tikzcd}
 & \prod_{i \in I}^{\succcurlyeq} \lambda_0(i) \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\
\lambda_0(i) & & \lambda_0(j) \arrow[ll, "\lambda_{ji}^{\succcurlyeq}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Y \arrow[dl, "\varpi_i"'] \arrow[dr, "\varpi_j"] & \\
\lambda_0(i) & & \lambda_0(j). \arrow[ll, "\lambda_{ji}^{\preccurlyeq}"]
\end{tikzcd}$$

(iii) If $\mathcal{G} := (Y, G)$ is a Bishop space and $\varpi_i : Y \to \lambda_0(i) \in \mathrm{Mor}(\mathcal{G}, \mathcal{F}_i)$, for every $i \in I$, such that if $i \preccurlyeq j$, the above right diagram commutes, then there is a unique function $h : Y \to \prod_{i \in I} \lambda_0(i) \in \mathrm{Mor}(\mathcal{G}, \underset{\leftarrow}{\mathrm{Lim}} \mathcal{F}_i)$ that makes the following diagrams commutative

$$
\begin{tikzcd}
 & Y \arrow[dl, "\varpi_i"'] \arrow[dr, "\varpi_j"] \arrow[dd, dashed, "h"] & \\
\lambda_0(i) & & \lambda_0(j), \arrow[ll, hook', "\lambda^{\succcurlyeq}_{ji}"] \\
 & \prod_{i \in I} \lambda_0(i) \arrow[ul, "\pi_i"] \arrow[ur, "\pi_j"'] &
\end{tikzcd}
$$

Theorem

Let $S^{\succcurlyeq} := (\lambda_0, \lambda_1^{\succcurlyeq}; \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ be a contravariant $(I, \preccurlyeq)$-spectrum with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ji}^{\succcurlyeq})_{(i,j) \in \preccurlyeq(I)}$, $T^{\succcurlyeq} := (\mu_0, \mu_1; \phi_0^{M^{\succcurlyeq}}, \phi_1^{M^{\succcurlyeq}})$ a contravariant $(I, \preccurlyeq)$-spectrum with Bishop spaces $(\mathcal{G}_i)_{i \in I}$ and Bishop morphisms $(\mu_{ji}^{\succcurlyeq})_{(i,j) \in \preccurlyeq(I)}$, and let $\Psi : S^{\succcurlyeq} \Rightarrow T^{\succcurlyeq}$.

(i) There is a unique function $\Psi_{\leftarrow} : \underset{\leftarrow}{\mathrm{Lim}} \lambda_0(i) \to \underset{\leftarrow}{\mathrm{Lim}} \mu_0(i)$ such that, for every $i \in I$, the following diagram commutes

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, "\Psi_i"] & \mu_0(i) \\
\underset{\leftarrow}{\mathrm{Lim}} \lambda_0(i) \arrow[u, "\pi_i^{S^{\succcurlyeq}}"] \arrow[r, dashed, "\Psi_{\leftarrow}"'] & \underset{\leftarrow}{\mathrm{Lim}} \mu_0(i). \arrow[u, "\pi_i^{T^{\succcurlyeq}}"']
\end{tikzcd}$$

(ii) If Ψ is continuous, then $\Psi_{\leftarrow} \in \mathrm{Mor}(\underset{\leftarrow}{\mathrm{Lim}} \mathcal{F}_i, \underset{\leftarrow}{\mathrm{Lim}} \mathcal{G}_i)$.

(iii) If Ψ_i is an injection, for every $i \in I$, then $\Psi_{\leftarrow}$ is an injection.

Proposition

Let $\mathcal{F} := (X, F), \mathcal{G} := (Y, G)$ *and* $\mathcal{H} := (Z, H)$ *be Bishop spaces, and let* $\lambda \in \mathrm{Mor}(\mathcal{G}, \mathcal{H}), \mu \in \mathrm{Mor}(\mathcal{H}, \mathcal{G})$*. If*

$$\lambda^{+} : \mathrm{Mor}(\mathcal{H}, \mathcal{F}) \to \mathrm{Mor}(\mathcal{G}, \mathcal{F}) \qquad \lambda^{+}(\phi) := \phi \circ \lambda,$$

$$\mu^{-} : \mathrm{Mor}(\mathcal{F}, \mathcal{H}) \to \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \qquad \mu^{-}(\theta) := \mu \circ \theta,$$

$$\begin{tikzcd}
Z \arrow[r, "\phi"] & X \arrow[r, "\theta"] \arrow[dr, "\mu \circ \theta"'] & Z \arrow[d, "\mu"] \\
Y \arrow[u, "\lambda"] \arrow[ur, "\phi \circ \lambda"] & & Y,
\end{tikzcd}$$

$$^{+} : \mathrm{Mor}(\mathcal{G}, \mathcal{H}) \to \mathrm{Mor}(\mathcal{H} \to \mathcal{F}, \mathcal{G} \to \mathcal{F}) \qquad \lambda \mapsto \lambda^{+},$$

$$^{-} : \mathrm{Mor}(\mathcal{H}, \mathcal{G}) \to \mathrm{Mor}(\mathcal{F} \to \mathcal{H}, \mathcal{F} \to \mathcal{G}) \qquad \mu \mapsto \mu^{-},$$

are in $\mathrm{Mor}\big(\mathcal{G} \to \mathcal{H}, (\mathcal{H} \to \mathcal{F}) \to (\mathcal{G} \to \mathcal{F})\big)$ *and in* $\mathrm{Mor}\big(\mathcal{H} \to \mathcal{G}, (\mathcal{F} \to \mathcal{H}) \to (\mathcal{F} \to \mathcal{G})\big)$*, respectively.*

Proposition

(A) Let $S^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ *and* $\mathcal{F} := (X, F)$.

(i) If $S^{\preccurlyeq} \to \mathcal{F} := (\mu_0, \mu_1^{\succcurlyeq}; \phi_0^{M^{\succcurlyeq}}, \phi_1^{M^{\succcurlyeq}})$, *where* $M^{\succcurlyeq} := (\mu_0, \mu_1^{\succcurlyeq})$ *is a contravariant direct family of sets over* $(I, \preccurlyeq)$ *with* $\mu_0(i) := \mathrm{Mor}(\mathcal{F}_i, \mathcal{F})$ *and*

$$\mu_1^{\succcurlyeq}(i,j) := \big(\mathrm{Mor}(\mathcal{F}_j, \mathcal{F}), \mathrm{Mor}(\mathcal{F}_i, \mathcal{F}), (\lambda_{ij}^{\preccurlyeq})^{+}\big),$$

and $\phi_0^{M^{\succcurlyeq}}(i) := F_i \to F$, $\phi_1^{M^{\succcurlyeq}}(i,j) := \big(F_i \to F, F_j \to F, [(\lambda_{ij}^{\preccurlyeq})^{+}]^{*}\big)$, *then* $S^{\preccurlyeq} \to \mathcal{F}$ *is a contravariant* $(I, \preccurlyeq)$*-spectrum with Bishop spaces* $(\mathrm{Mor}(\mathcal{F}_i, \mathcal{F}))_{i \in I}$ *and Bishop morphisms* $\big((\lambda_{ij}^{\preccurlyeq})^{+}\big)_{(i,j) \in \preccurlyeq(I)}$.

(ii) If $\mathcal{F} \to S^{\preccurlyeq} := (\nu_0, \nu_1^{\preccurlyeq}; \phi_0^{N^{\preccurlyeq}}, \phi_1^{N^{\preccurlyeq}})$, *where* $N^{\preccurlyeq} := (\nu_0, \nu_1^{\preccurlyeq})$ *is a direct family of sets over* $(I, \preccurlyeq)$ *with* $\nu_0(i) := \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i)$ *and*

$$\nu_1^{\preccurlyeq}(i,j) := \big(\mathrm{Mor}(\mathcal{F}, \mathcal{F}_i), \mathrm{Mor}(\mathcal{F}, \mathcal{F}_j), (\lambda_{ij}^{\preccurlyeq})^{-}\big),$$

and if $\phi_0^{N^{\preccurlyeq}}(i) := F \to F_i$, $\phi_1^{N^{\preccurlyeq}}(i,j) := \big(F \to F_j, F \to F_i, [(\lambda_{ij}^{\preccurlyeq})^{-}]^{*}\big)$, *then* $\mathcal{F} \to S^{\preccurlyeq}$ *is a covariant* $(I, \preccurlyeq)$*-spectrum with Bishop spaces* $(\mathrm{Mor}(\mathcal{F}, \mathcal{F}_i))_{i \in I}$ *and Bishop morphisms* $\big((\lambda_{ij}^{\preccurlyeq})^{-}\big)_{(i,j) \in \preccurlyeq(I)}$.

Proposition

(B) Let $S^{\succcurlyeq} := (\lambda_0, \lambda_1^{\succcurlyeq}; \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ *and* $\mathcal{F} := (X, F)$.
(i) If $S^{\succcurlyeq} \to \mathcal{F} := (\mu_0, \mu_1^{\preccurlyeq}; \phi_0^{M^{\preccurlyeq}}, \phi_1^{M^{\preccurlyeq}})$*, where* $M^{\preccurlyeq} := (\mu_0, \mu_1^{\preccurlyeq})$ *is a direct family of sets over* $(I, \preccurlyeq)$ *with* $\mu_0(i) := \mathrm{Mor}(\mathcal{F}_i, \mathcal{F})$ *and*

$$\mu_1^{\preccurlyeq}(i,j) := \big(\mathrm{Mor}(\mathcal{F}_i, \mathcal{F}), \mathrm{Mor}(\mathcal{F}_j, \mathcal{F}), (\lambda_{ji}^{\succcurlyeq})^+\big),$$

and $\phi_0^{M^{\preccurlyeq}}(i) := F_i \to F$, $\phi_1^{M^{\preccurlyeq}}(i,j) := \big(F_j \to F, F_i \to F, [(\lambda_{ji}^{\succcurlyeq})^+]^*\big)$,
then $S^{\succcurlyeq} \to \mathcal{F}$ *is an* $(I, \preccurlyeq)$*-spectrum with Bishop spaces* $(\mathrm{Mor}(\mathcal{F}_i, \mathcal{F}))_{i \in I}$ *and Bishop morphisms* $\big((\lambda_{ji}^{\succcurlyeq})^+\big)_{(i,j) \in \preccurlyeq(I)}$.
(ii) If $\mathcal{F} \to S^{\succcurlyeq} := (\nu_0, \nu_1^{\succcurlyeq}; \phi_0^{N^{\succcurlyeq}}, \phi_1^{N^{\succcurlyeq}})$*, where* $N^{\succcurlyeq} := (\nu_0, \nu_1^{\succcurlyeq})$ *is a contravariant direct family of sets over* $(I, \preccurlyeq)$ *with* $\nu_0(i) := \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i)$ *and*

$$\nu_1^{\succcurlyeq}(i,j) := \big(\mathrm{Mor}(\mathcal{F}, \mathcal{F}_j), \mathrm{Mor}(\mathcal{F}, \mathcal{F}_i), (\lambda_{ji}^{\succcurlyeq})^-\big),$$

and $\phi_0^{N^{\succcurlyeq}}(i) := F \to F_i$, $\phi_1^{N^{\succcurlyeq}}(i,j) := \big(F \to F_i, F \to F_j, [(\lambda_{ji}^{\succcurlyeq})^-]^*\big)$,
then $\mathcal{F} \to S^{\preccurlyeq}$ *is a contravariant* $(I, \preccurlyeq)$*-spectrum with Bishop spaces* $(\mathrm{Mor}(\mathcal{F}, \mathcal{F}_i))_{i \in I}$ *and Bishop morphisms* $\big((\lambda_{ji}^{\preccurlyeq})^-\big)_{(i,j) \in \preccurlyeq(I)}$.

If $S^{\succcurlyeq} := (\lambda_0, \lambda_1^{\succcurlyeq}; \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ a contravariant direct spectrum over $(I, \preccurlyeq)$ with Bishop spaces $\big(F_i = \bigvee F_{0i}\big)_{i \in I}$, then

$$\prod_{i \in I}^{\succcurlyeq} F_i = \bigvee_{i \in I}^{f \in F_{0i}} \big(f \circ \pi_i^{\Lambda^{\succcurlyeq}}\big).$$

Theorem (Duality principle)

Let $(I, \preccurlyeq)$ be a directed set, $S^{\preccurlyeq} := (\lambda_0, \lambda_1^{\preccurlyeq}; \phi_0^{\Lambda^{\preccurlyeq}}, \phi_1^{\Lambda^{\preccurlyeq}})$ an $(I, \preccurlyeq)$-direct spectrum with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ij}^{\preccurlyeq})_{(i,j) \in \preccurlyeq(I)}$. If $\mathcal{F} := (X, F)$ is a Bishop space and $S^{\preccurlyeq} \to \mathcal{F} := (\mu_0, \mu_1^{\succcurlyeq}, \phi_0^{M^{\succcurlyeq}}, \phi_1^{M^{\succcurlyeq}})$ is the previous $A(i)$ contravariant direct spectrum over $(I, \preccurlyeq)$, then

$$\underset{\leftarrow}{\mathrm{Lim}}(\mathcal{F}_i \to \mathcal{F}) \simeq [(\underset{\rightarrow}{\mathrm{Lim}}\mathcal{F}_i) \to \mathcal{F}].$$

Theorem

Let $(I, \preccurlyeq)$ be a directed set, $S^{\succcurlyeq} := (\lambda_0, \lambda_1^{\succcurlyeq}; \phi_0^{\Lambda^{\succcurlyeq}}, \phi_1^{\Lambda^{\succcurlyeq}})$ a contravariant direct spectrum over $(I, \preccurlyeq)$ with Bishop spaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\lambda_{ji}^{\preccurlyeq})_{(i,j) \in \preccurlyeq(I)}$. If $\mathcal{F} := (X, F)$ is a Bishop space and $\mathcal{F} \to S^{\succcurlyeq} := (\nu_0, \nu_1^{\succcurlyeq}; \phi_0^{\succcurlyeq}, \phi_1^{N^{\succcurlyeq}})$ is the contravariant direct spectrum over $(I, \preccurlyeq)$, defined above in the previous Proposition (B)(ii), then

$$\underset{\leftarrow}{\mathrm{Lim}}(\mathcal{F} \to \mathcal{F}_i) \simeq [\mathcal{F} \to \underset{\leftarrow}{\mathrm{Lim}} \mathcal{F}_i].$$

- We tried to show how some fundamental notions of Bishop's set theory, formulated in an explicit way within the reconstruction BST of Bishop's system, can be applied to the constructive topology of Bishop spaces, and especially in the theory of limits of Bishop spaces.
- The definition of e.g., the sum Bishop topology on the corresponding direct sum shows a harmonious relation between Bishop sets and Bishop spaces.
- We can define generalised I-families of sets, or generalised families of sets over a directed set $(I, \preccurlyeq)$, where more than one transport maps from $\lambda_0(i)$ to $\lambda_0(j)$ are permitted.
- One can study the direct spectra of Bishop subspaces.

A family of subsets of X indexed by I is a triple $\Lambda_X := (\lambda_0, \mathcal{E}, \lambda_1)$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, and

$$\mathcal{E} : \bigcurlywedge_{i \in I} \mathbb{F}\big(\lambda_0(i), X\big), \qquad \mathcal{E}(i) := \mathcal{E}_i, \quad i \in I,$$

$$\lambda_1 : \bigcurlywedge_{(i,j) \in D(I)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big), \quad \lambda_1(i,j) := \lambda_{ij}, \quad (i,j) \in D(I)$$

- $\mathcal{E}_i : \lambda_0(i) \to X$ is an embedding,
- $\lambda_{ii} := \mathrm{id}_{\lambda_0(i)}$,
- $\mathcal{E}_i = \mathcal{E}_j \circ \lambda_{ij}$ and $\mathcal{E}_j = \mathcal{E}_i \circ \lambda_{ji}$

$$\begin{tikzcd}
\lambda_0(i) \arrow[rr, hook, bend left, "\lambda_{ij}"] \arrow[dr, hook, "\mathcal{E}_i"'] & & \lambda_0(j) \arrow[ll, hook, bend left, "\lambda_{ji}"] \arrow[dl, hook', "\mathcal{E}_j"] \\
& X &
\end{tikzcd}$$

- The internal equality $(\lambda_{ij}, \lambda_{ji}) : \lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$ implies the external equality $(\lambda_{ij}, \lambda_{ji}) : \lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$.

$$\mathcal{E}_k \circ (\lambda_{jk} \circ \lambda_{ij}) = (\mathcal{E}_k \circ \lambda_{jk}) \circ \lambda_{ij} = \mathcal{E}_j \circ \lambda_{ij} = \mathcal{E}_i \quad \& \quad \mathcal{E}_k \circ \lambda_{ik} = \mathcal{E}_i$$

$$\begin{tikzcd}
\lambda_0(i) \arrow[r, hook, "\lambda_{ij}"'] \arrow[rr, hook, bend left, "\lambda_{ik}"] \arrow[dr, hook, "\mathcal{E}_i"'] & \lambda_0(j) \arrow[r, hook, "\lambda_{jk}"'] \arrow[d, hook, "\mathcal{E}_j"'] & \lambda_0(k) \arrow[dl, hook', "\mathcal{E}_k"] \\
& X, &
\end{tikzcd}$$

hence $\mathcal{E}_k \circ (\lambda_{jk} \circ \lambda_{ij}) = \mathcal{E}_k \circ \lambda_{ik}$, and since $\mathcal{E}_k$ is an embedding, we get $\lambda_{jk} \circ \lambda_{ij} = \lambda_{ik}$.

If $\Lambda_X := (\lambda_0, \mathcal{E}, \lambda_1)$ and $M_X := (\mu_0, E, \mu_1)$ are I-families of subsets of X, a family of subsets-map from Λ_X to M_X is a dependent assignment routine

$$\Psi : \bigcurlywedge_{i \in I} \mathbb{F}\big(\lambda_0(i), \mu_0(i)\big), \qquad \Psi(i) := \Psi_i, \quad i \in I,$$

such that for every $i \in I$ the following diagram commutes

$$\begin{tikzcd} \lambda_0(i) \arrow[rr, "\Psi_i"] \arrow[dr, hook, "\mathcal{E}_i"'] & & \mu_0(i) \arrow[dl, hook', "E_i"] \\ & X. & \end{tikzcd}$$

- If $\Psi : \Lambda_X \Rightarrow M_X$, then $\Psi : (\lambda_0, \lambda_1) \Rightarrow (\mu_0, \mu_1)$.

  If $x \in \lambda_0(i)$, then

$$E_j\big(\Psi_j(\lambda_{ij}(x))\big) := (E_j \circ \Psi_j)\big(\lambda_{ij}(x)\big) = \mathcal{E}_j\big(\lambda_{ij}(x)\big) = \mathcal{E}_i(x)$$

$$= (E_i \circ \Psi_i)(x) = E_j\big(\mu_{ij}(\Psi_i(x))\big),$$

  hence $\Psi_j(\lambda_{ij}(x) = \mu_{ij}(\Psi_i(x))$.

Let $\Lambda_X := (\lambda_0, \mathcal{E}, \lambda_1)$ be an I-family of subsets of X. The interior union of Λ_X is the totality $\sum_{i \in I} \lambda_0(i)$, which we denote in this case by $\bigcup_{i \in I} \lambda_0(i)$.

Let the assignment routine $\varepsilon : \bigcup_{i \in I} \lambda_0(i) \rightsquigarrow X$

$$(i, x) \mapsto \mathcal{E}_i(x),$$

$$(i, x) =_{\bigcup_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow \varepsilon(i, x) =_X \varepsilon(j, y) :\Leftrightarrow \mathcal{E}_i(x) =_X \mathcal{E}_j(y).$$

$$\left(\bigcup_{i \in I} \lambda_0(i), \varepsilon \right) \subseteq X.$$

Let $\Lambda_X := (\lambda_0, \mathcal{E}, \lambda_1)$ be an I-family of subsets of X, where I is inhabited by some element i_0. The intersection $\bigcap_{i \in I} \lambda_0(i)$ of Λ_X is the totality defined by

$$\Phi \in \bigcap_{i \in I} \lambda_0(i) :\Leftrightarrow \Phi : \bigcurlywedge_{i \in I} \lambda_0(i) \ \& \ \forall_{i, i' \in I}\big(\mathcal{E}_i(\Phi_i) =_X \mathcal{E}_{i'}(\Phi_{i'})\big).$$

Let the assignment routine $e : \bigcap_{i \in I} \lambda_0(i) \rightsquigarrow X$

$$e(\Phi) := \mathcal{E}_{i_0}\big(\Phi_{i_0}\big).$$

$$\Phi =_{\bigcap_{i \in I} \lambda_0(i)} \Theta :\Leftrightarrow e(\Phi) =_X e(\Theta) :\Leftrightarrow \mathcal{E}_{i_0}\big(\Phi_{i_0}\big) =_X \mathcal{E}_{i_0}\big(\Theta_{i_0}\big).$$

$$\left(\bigcap_{i \in I} \lambda_0(i), e\right) \subseteq X.$$

Let $(I, \preccurlyeq)$ be a directed set, and X a set. A direct family of subsets of X indexed by I is a triple $\Lambda_X^{\preccurlyeq} := (\lambda_0, \mathcal{E}, \lambda_1^{\preccurlyeq})$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$,

$$\mathcal{E} : \bigcurlywedge_{i \in I} \mathbb{F}\big(\lambda_0(i), X\big), \qquad \mathcal{E}(i) := \mathcal{E}_i, \quad i \in I,$$

$$\lambda_1^{\preccurlyeq} : \bigcurlywedge_{(i,j) \in \preccurlyeq(I)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big), \quad \lambda_1^{\preccurlyeq}(i,j) := \lambda_{ij}^{\preccurlyeq}, \quad (i,j) \in \preccurlyeq(I),$$

such that the following conditions hold:

(a) For every $i \in I$, the function $\mathcal{E}_i : \lambda_0(i) \to X$ is an embedding.

(b) For every $i \in I$, we have that $\lambda_{ii} := \mathrm{id}_{\lambda_0(i)}$.

(c) For every $(i,j) \in \preccurlyeq(I)$ we have that $\mathcal{E}_i = \mathcal{E}_j \circ \lambda_{ij}^{\preccurlyeq}$

$$\begin{tikzcd}
\lambda_0(i) \arrow[rr, hook, "\lambda_{ij}^{\preccurlyeq}"] \arrow[dr, hook, "\mathcal{E}_i"'] & & \lambda_0(j) \arrow[dl, hook', "\mathcal{E}_j"] \\
& X. &
\end{tikzcd}$$

Let $\mathcal{F} := (X, F)$ be a Bishop space, $(I, \preccurlyeq)$ a directed set, and let $\Lambda_X^{\preccurlyeq} := (\lambda_0, \mathcal{E}, \lambda_1^{\preccurlyeq})$ be an $(I, \preccurlyeq)$-family of subsets of X. For every $i \in I$ let the relative Bishop space $\mathcal{F}_i := \mathcal{F}_{|\lambda_0(i)} := (\lambda_0(i), F_i)$ i.e.,

$$F_i := F_{|\lambda_0(i)} := \bigvee_{f \in F} f \circ \mathcal{E}_i.$$

Let $\phi_0^{\Lambda_X}(i) := F_i$, and $\phi_1^{\Lambda_X} : \curlywedge_{(i,j) \in \preccurlyeq(I)} \mathbb{F}(F_i, \mathbb{F}_j)$, where $\phi_1^{\Lambda_X}(i,j) := \lambda_{ji}^*$, for every $(i,j) \in \preccurlyeq (I)$. We call the structure

$$S_X^{\preccurlyeq} := (\lambda_0, \mathcal{E}, \lambda_1^{\preccurlyeq}; F; \phi_0^{\Lambda_X^{\preccurlyeq}}, \phi_1^{\Lambda_X^{\preccurlyeq}}),$$

direct spectrum over $(I, \preccurlyeq)$ with Bishop subspaces $(\mathcal{F}_i)_{i \in I}$ and Bishop morphisms $(\mathcal{E}_i)_{i \in I}$.

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