"Neutral" Models of Constructive Mathematics

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Often in semantics one builds a new model $\mathcal{E}$ over a *ground model* $\mathcal{S}$ as e.g. in topological semantics, realizability, topos theory...

and there is a so-called *constant objects* (CO) functor

$$F : \mathcal{S} \to \mathcal{E}$$

describing how the ground model $\mathcal{S}$ sits within the new model $\mathcal{E}$.

Typically this F faithfully represents the construction of $\mathcal{E}$ from $\mathcal{S}$.

Iteration of constructions as composition of CO functors.

Via "Artin Glueing" we obtain a new model $\mathsf{Gl}(F) = \mathcal{E}{\downarrow}F$ together with a logical functor

$$P_F = \partial_1 = \mathsf{cod} : \mathcal{E}{\downarrow}F \to \mathcal{S}$$

which, therefore, is consistent with $\mathcal{S}$ which often is **Set**!

Heyting (Boolean) Valued Sets

Let A be a complete Heyting (or boolean) algebra in a base topos $\mathcal{S}$ then the topos $Sh_{\mathcal{S}}(A)$ of sheaves over A contains the base $\mathcal{S}$ via $F : \mathcal{S} \to \mathcal{E}$ sending I to the "constant sheaf" with value I. Thinking of "$\mathcal{E}$ as A-valued sets" we have $F(I) = (I, eq_I)$ where $eq_I(i,j) = \bigvee\{1_A \mid i = j\}$.

The CO functor F preserves finite limits, has a right adjoint U and every $X \in \mathcal{E}$ appears as subquotient of some FI.

Such adjunctions $F \dashv U : \mathcal{E} \to \mathbf{Set}$ are called "localic geometric morphisms" since the latter condition says that subobjects of $1_{\mathcal{E}}$ generate. Under these assumptions $\mathcal{E}$ is equivalent to $Sh_{\mathcal{S}}(U\Omega_{\mathcal{E}})$

Since maps maps $I \to U\Omega_{\mathcal{E}}$ correspond to maps $FI \to \Omega_{\mathcal{E}}$, i.e. subobjects of FI, the *externalization* of $U\Omega_{\mathcal{E}}$ is given by $F^*\mathrm{Sub}_{\mathcal{E}}$ (where $\mathrm{Sub}_{\mathcal{E}}$ is the subobject fibration of $\mathcal{E}$).

The Moens-Jibladze Correspondence (1)

If $F : \mathcal{S} \to \mathcal{E}$ is a finite limit preserving functor between toposes we may consider the (Grothendieck) fibration P_F as in

$$\begin{array}{ccc} \mathcal{E}{\downarrow}F & \longrightarrow & \mathcal{E}{\downarrow}\mathcal{E} \\ {\scriptstyle P_F}\downarrow & & \downarrow{\scriptstyle P_{\mathcal{E}}} \\ \mathcal{S} & \xrightarrow[F]{} & \mathcal{E} \end{array}$$

where $P_{\mathcal{E}}$ (and thus also P_F) is the codomain functor.
All fibers of P_F are toposes and all reindexing functors are logical (i.e. preserve finite limits, exponentials and subobject classifiers) and P_F has internal sums (i.e. P_F is a cofibration where cocartesian arrows are stable under pullbacks along cartesian arrows in $\mathcal{E}$).

The Moens-Jibladze Correspondence (2)

Such fibrations $P : \mathcal{X} \to \mathcal{S}$ are called *fibered toposes with internal sums*.

M. Jibladze has shown that internal sums are necessarily *stable and disjoint* from which it follows by Moens's Theorem that $P : \mathcal{X} \to \mathcal{S}$ is equivalent to P_F where $F : \mathcal{S} \to \mathcal{E} = P(1)$ sends $u : J \to I$ to the unique vertical arrow Fu rendering the diagram

$$\begin{array}{ccc} 1_J & \xrightarrow[\text{cocart.}]{\varphi_J} & FJ \\ {\scriptstyle 1_u}\downarrow & & \downarrow{\scriptstyle Fu} \\ 1_I & \xrightarrow[\varphi_I]{\text{cocart.}} & FI \end{array}$$

commutative. Up to equivalence this F is determined by P, informally speaking it sends $I \in \mathcal{S}$ to $\coprod_I 1_I$.

Properties of P_F in terms of properties of F

Further fibrational properties of P_F can be reformulated as elementary properties of F as follows

1. P_F is locally small iff F has a right adjoint U
2. P_F has a small generating family iff there is a *bound* $B \in \mathcal{E}$ such that every $X \in \mathcal{E}$ appears as subquotient of some $B \times FI$.

In particular, P_F is a localic topos fibered over $\mathcal{S}$ iff P_F is locally small and $F \dashv U$ is bounded by $1_{\mathcal{E}}$.

Triposes as Generalized Localic Toposes (1)

A **tripos** over a base topos $\mathcal{S}$ is a functor F from $\mathcal{S}$ to a topos $\mathcal{E}$ such that

1. F preserves finite limits and
2. every $A \in \mathcal{E}$ appears as subquotient of FI for some $I \in \mathcal{S}$.

A tripos $F : \mathcal{S} \to \mathcal{E}$ is **strong** iff F preserves also epis (which trivially holds if $\mathcal{S}$ is **Set** since there all epis are split!).
A tripos $F : \mathcal{S} \to \mathcal{E}$ is **traditional** iff there is a subobject $\tau : T \rightarrowtail \Sigma$ such that every mono $m : P \rightarrowtail FI$ fits into a pullback

$$\begin{array}{ccc} P & \longrightarrow & T \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow{\scriptstyle \tau} \\ FI & \xrightarrow[Fp]{} & F\Sigma \end{array}$$

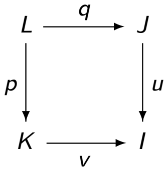
for some (typically not unique) $p : I \to \Sigma$.

Triposes as Generalized Localic Toposes (2)

With every traditional tripos $F : \mathcal{S} \to \mathcal{E}$ one can associate the fibered poset $\mathscr{P}_F = F^*\mathrm{Sub}_{\mathcal{E}}$ validating the conditions

1. $\mathscr{P}_F$ is a fibration of pre-Heyting-algebras
2. for every u in the base the reindexing map $u^* = \mathscr{P}_F(u)$ has both adjoints $\exists_u \dashv u^* \dashv \forall_u$ (as a map of preorders) validating the (Beck-)Chevalley condition[1]
3. there is a generic $\tau \in \mathscr{P}_F(\Sigma)$ such that every $\varphi \in \mathscr{P}_F(I)$ is isomorphic to $p^*\tau$ for some $p : I \to \Sigma$.

[1] i.e. we have $v^*\exists_u \dashv\vdash \exists_p q^*$ and $v^*\forall_u \dashv\vdash \forall_p q^*$ for all pullbacks



Triposes as Generalized Localic Toposes (3)

If F is just a tripos then the third condition has to be weakened as follows:

for very $I \in \mathcal{S}$ there is a $P(I)$ in $\mathcal{S}$ and ε_I in $\mathscr{P}_F(I \times P(I))$ such that for every ρ in $\mathscr{P}_F(I \times J)$

$$\text{(Comp)} \quad \forall j \in J. \exists p \in P(I). \forall i \in I.\, \rho(i,j) \leftrightarrow i \, \varepsilon_I \, p$$

holds in the logic of $\mathscr{P}_F$.

This is the usual *comprehension principle* for HOL.
Its Skolemized (and thus stronger) version is equivalent to the existence of a generic subterminal $\tau : T \rightarrowtail F\Sigma$ (where Σ is $P(1)$).

But the logic of the tripos does not validate extensionality for predicates, i.e. p is not uniquely determined by j.

Triposes as Generalized Localic Toposes (4)

For triposes $F : \mathcal{S} \to \mathcal{E}$ the CO functor $\mathcal{S} \to \mathcal{S}[\mathscr{P}_F]$ is equivalent to F and a tripos $\mathscr{P}$ is equivalent to $\mathscr{P}_F$ where F is the CO functor $\mathcal{S} \to \mathcal{S}[\mathscr{P}]$ as shown in Pitts's Thesis.

Here $\mathcal{S}[\mathscr{P}]$ is obtained from $\mathscr{P}$ by "adding quotients" defining morphisms as functional relations.
The CO functor $\mathcal{S} \to \mathcal{S}[\mathscr{P}]$ sends I to (I, eq_I) where $eq_I = \exists_{\delta_I} \top_I$.

Uniqueness of Constant Objects Functors?

Are triposes $F_1, F_2 : \mathcal{S} \to \mathcal{E}$ necessarily equivalent?

The answer is in general NO if $\mathcal{S}$ is not equal to **Set** since for sober (e.g. Hausdorff spaces) X and Y there are as many localic geometric morphism $\mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ as there are continuous maps from Y to X.

For all natural numbers $n > 0$ the functor

$$F_n : \mathbf{Set} \to \mathbf{Set} : I \mapsto I^n$$

is a tripos. But F_n and F_m are equivalent iff $n = m$.

Alas, the question is open for traditional triposes over **Set** since in the above counterexample only F_1 is a traditional tripos.

Question even open for localic and realizability toposes!

Already in [HJP80] where triposes were introduced it was asked whether localic toposes $\mathrm{Sh}(A)$ over **Set** may be induced by traditional triposes whose constant objects functor is not equivalent to $\Delta : \mathbf{Set} \to \mathrm{Sh}(A)$.

Maybe we get such examples via classical realizability? Krivine's criterion (absence of "parallel or") for a realizability algebra only guarantees that the associated tripos is not localic but not that the induced topos is not localic...e.g. possibly **Set**.

Also realizability toposes $\mathrm{RT}(\mathcal{A})$ over **Set** could be induced by triposes whose constant objects functor is not equivalent to $\nabla : \mathbf{Set} \to \mathrm{RT}(\mathcal{A})$.

Non-Localic Grothendieck Toposes from Triposes over **Set**

If $\mathcal{E}$ is the topos of *reflexive graphs* $\mathbf{Set}^{\Delta_1^{\mathrm{op}}}$ or the topos $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ of *simplicial sets* then $\nabla : \mathbf{Set} \to \mathcal{E}$ (right adjoint to $\Gamma = \mathcal{E}(1, -)$) is a (strong) tripos which, however, is not traditional.

Every reflexive graph may be covered by a subobject of some $\nabla(S)$!

Possibly, this also holds for the topos of cubical sets $\mathbf{Set}^{\Box^{\mathrm{op}}}$ (where $\Box$ is the full subcat of **Poset** on finite powers of the ordinal 2)?

Neutral Models via Glueing

Together with P. Lietz I showed that the extensional realizability topos **Ext** doesn't validated Ishihara's $\mathrm{BD}_{\mathbb{N}}$.
But **Ext** validates a negative form of Church's Thesis, namely

$$\forall f : \mathbb{N} \to \mathbb{N}. \neg\neg\exists e : \mathbb{N}.\, f = \{e\}$$

and thus is not conservative over **Set**.

But for every finite limit preserving functor $F : \mathcal{S} \to \mathcal{E}$ between toposes the comma category $\mathcal{E}{\downarrow}F$ is a topos and the functor $P_F = \partial_1 = \mathsf{cod} : \mathcal{E}{\downarrow}F \to \mathcal{S}$ is logical and has full and faithful left and right adjoints sending $I \in \mathcal{S}$ to $0 \to FI$ and id_{FI}, respectively.

For triposes $F : \mathbf{Set} \to \mathcal{E}$ the comma category $\mathcal{E}{\downarrow}F$ is a topos and $P_F = \mathsf{cod} : \mathcal{E}{\downarrow}F \to \mathbf{Set}$ is logical.

Thus $\mathcal{E}{\downarrow}F$ only validates sentences which hold in **Set** and thus is a **neutral** model of constructive mathematics.

Summary

- Ground models are typically not unique! (Since **Set** is induced by infinitely many non-equivalent triposes over **Set**).
- Question open for traditional triposes over **Set** even for localic and realizability toposes though there are canonical candidates Δ and ∇, respectively. But are these the only possibilities?
- Triposes F over **Set** via "Artin Glueing" give rise to **neutral** models $\mathcal{E}{\downarrow}F$ since $P_F = \mathsf{cod} : \mathcal{E}{\downarrow}F \to \mathbf{Set}$ is logical.
- With a bit of luck $\mathcal{E}{\downarrow}F$ preserves some of the weaknesses of $\mathcal{E}$, e.g. doesn't validate FAN, $\mathrm{BD}_{\mathbb{N}}$, etc.

Analogue of cHa's for traditional strong triposes

A. Miquel has introduced a notion of *implicative algebra* and shown that every such i.a. $\mathcal{A}$ gives rise to a tripos $\mathscr{P}^{\mathcal{A}}$ over **Set** and every traditional tripos over **Set** arises this way as $\Delta_{\mathcal{A}} : \mathbf{Set} \to \mathbf{Set}[\mathscr{P}^{\mathcal{A}}]$.

This generalizes to base toposes $\mathcal{S}$ with nno: every traditional strong tripos $F : \mathcal{S} \to \mathcal{E}$ is equivalent to $\Delta_{\mathcal{A}} : \mathcal{S} \to \mathcal{S}[\mathscr{P}^{\mathcal{A}}]$ for some i.a. $\mathcal{A}$ in $\mathcal{S}$ where $\Delta_{\mathcal{A}}(I) = (I, eq_I)$.

Implicative Structures

An *implicative structure* is a complete lattice $\mathcal{A} = (A, \leq)$ together with an implication operation $\rightarrow : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \rightarrow \mathcal{A}$ such that $y \rightarrow \bigwedge X = \bigwedge_{x \in X} (y \rightarrow x)$ for all $y \in \mathcal{A}$ and $X \subseteq \mathcal{A}$.

Thus $y \rightarrow (-)$ has a left adjoint $(-)y$ given by

$$xy = \bigwedge \{ z \mid x \leq y \rightarrow z \}$$

Then $K_{\mathcal{A}} = \bigwedge_{x,y \in \mathcal{A}} x {\rightarrow} y {\rightarrow} x$ and $S_{\mathcal{A}} = \bigwedge_{x,y,z \in \mathcal{A}} (x {\rightarrow} y {\rightarrow} z) {\rightarrow} (x {\rightarrow} y) {\rightarrow} x {\rightarrow} z$ are elements of $\mathcal{A}$ for which we have

$$K_{\mathcal{A}} xy \leq x \qquad \textit{and} \qquad S_{\mathcal{A}} xyz = xz(yz)$$

Implicative Algebras

A *separator* in an implicative structure $(\mathcal{A}, \to)$ is an upward closed subset $\mathcal{S}$ of $\mathcal{A}$ such that $K_{\mathcal{A}}, S_{\mathcal{A}} \in \mathcal{S}$ and $\mathcal{S}$ is closed under *modus ponens*, i.e. $b \in \mathcal{S}$ whenever $a \in \mathcal{S}$ and $a \to b \in \mathcal{S}$.
An *implicative algebra* is a triple $(\mathcal{A}, \to, \mathcal{S})$ such that $(\mathcal{A}, \to)$ is an implicative structure and $\mathcal{S}$ is a separator in $(\mathcal{A}, \to)$.
With every implicative algebra $\mathcal{A}$ one associates a **Set**-based tripos $\mathscr{P}^{\mathcal{A}}$ where $\mathscr{P}^{\mathcal{A}}(I)$ is the preorder $\vdash_I$ on $\mathcal{A}^I$ defined as

$$\varphi \vdash_I \psi \quad \text{iff} \quad \bigwedge_{i \in I} (\varphi_i \to \psi_i) \in \mathcal{S}$$

and reindexing is given by precomposition.