“Neutral” Models of Constructive Mathematics

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Often in semantics one builds a new model $E$ over a *ground model* $S$ as e.g. in topological semantics, realizability, topos theory... and there is a so-called *constant objects* (CO) functor

$$F : S \to E$$

describing how the ground model $S$ sits within the new model $E$. Typically this $F$ faithfully represents the construction of $E$ from $S$.

Iteration of constructions as composition of CO functors.

Via “Artin Glueing” we obtain a new model $\text{Gl}(F) = E \downarrow F$ together with a logical functor

$$P_F = \partial_1 = \text{cod} : E \downarrow F \to S$$

which, therefore, is consistent with $S$ which often is $\textbf{Set}$!
Let $A$ be a complete Heyting (or boolean) algebra in a base topos $S$ then the topos $Sh_S(A)$ of sheaves over $A$ contains the base $S$ via $F : S \to \mathcal{E}$ sending $I$ to the “constant sheaf” with value $I$. Thinking of $\mathcal{E}$ as $A$-valued sets” we have $F(I) = (I, eq_I)$ where $eq_I(i, j) = \bigvee \{1_A \mid i = j\}$.

The CO functor $F$ preserves finite limits, has a right adjoint $U$ and every $X \in \mathcal{E}$ appears as subquotient of some $FI$.

Such adjunctions $F \dashv U : \mathcal{E} \to \textbf{Set}$ are called ”localic geometric morphisms” since the latter condition says that subobjects of $1_\mathcal{E}$ generate. Under these assumptions $\mathcal{E}$ is equivalent to $Sh_S(U\Omega_\mathcal{E})$

Since maps $I \to U\Omega_\mathcal{E}$ correspond to maps $FI \to \Omega_\mathcal{E}$, i.e. subobjects of $FI$, the externalization of $U\Omega_\mathcal{E}$ is given by $F^*\text{Sub}_\mathcal{E}$ (where Sub$_\mathcal{E}$ is the subobject fibration of $\mathcal{E}$).
If $F : S \to \mathcal{E}$ is a finite limit preserving functor between toposes we may consider the (Grothendieck) fibration $P_F$ as in

\[
\begin{array}{ccc}
\mathcal{E}_{\downarrow F} & \to & \mathcal{E}_{\downarrow \mathcal{E}} \\
\downarrow P_F & & \downarrow P_{\mathcal{E}} \\
S & \to & \mathcal{E}
\end{array}
\]

where $P_{\mathcal{E}}$ (and thus also $P_F$) is the codomain functor. All fibers of $P_F$ are toposes and all reindexing functors are logical (i.e. preserve finite limits, exponentials and subobject classifiers) and $P_F$ has internal sums (i.e. $P_F$ is a cofibration where cocartesian arrows are stable under pullbacks along cartesian arrows in $\mathcal{E}$).
Such fibrations $P : \mathcal{X} \to S$ are called *fibered toposes with internal sums*. M. Jibladze has shown that internal sums are necessarily *stable and disjoint* from which it follows by Moens’s Theorem that $P : \mathcal{X} \to S$ is equivalent to $P_F$ where $F : S \to \mathcal{E} = P(1)$ sends $u : J \to I$ to the unique vertical arrow $Fu$ rendering the diagram

\[
\begin{array}{ccc}
1_J & \xrightarrow{\varphi_J} & FJ \\
\downarrow{\text{cocart.}} & & \downarrow{\text{cocart.}} \\
1_U & \xrightarrow{\varphi_I} & FI \\
\end{array}
\]

commutative. Up to equivalence this $F$ is determined by $P$, informally speaking it sends $I \in S$ to $\bigsqcup_I 1_I$. 

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**The Moens-Jibladze Correspondence (2)**

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Further fibrational properties of $P_F$ can be reformulated as elementary properties of $F$ as follows:

1. $P_F$ is locally small iff $F$ has a right adjoint $U$.
2. $P_F$ has a small generating family iff there is a bound $B \in \mathcal{E}$ such that every $X \in \mathcal{E}$ appears as subquotient of some $B \times F I$.

In particular, $P_F$ is a localic topos fibered over $S$ iff $P_F$ is locally small and $F \downarrow U$ is bounded by $1_\mathcal{E}$. 

Streicher, "Neutral" Models of Constructive Mathematics
A **tripos** over a base topos $S$ is a functor $F$ from $S$ to a topos $E$ such that

1. $F$ preserves finite limits and
2. every $A \in E$ appears as subquotient of $FI$ for some $I \in S$.

A tripos $F : S \to E$ is **strong** iff $F$ preserves also epis (which trivially holds if $S$ is **Set** since there all epis are split!).

A tripos $F : S \to E$ is **traditional** iff there is a subobject $\tau : T \rightarrowtail \Sigma$ such that every mono $m : P \rightarrowtail FI$ fits into a pullback

$$
\begin{array}{ccc}
P & \rightarrow & T \\
m \downarrow & & \downarrow \tau \\
FI & \rightarrow & F\Sigma \\
\end{array}
$$

for some (typically not unique) $p : I \to \Sigma$. 

**Ref:** Streicher "Neutral" Models of Constructive Mathematics
With every traditional tripos $F : S \to \mathcal{E}$ one can associate the fibered poset $\mathcal{P}_F = F^* \text{Sub}_\mathcal{E}$ validating the conditions

1. $\mathcal{P}_F$ is a fibration of pre-Heyting-algebras

2. for every $u$ in the base the reindexing map $u^* = \mathcal{P}_F(u)$ has both adjoints $\exists_u \vdash u^* \dashv \forall_u$ (as a map of preorders) validating the (Beck-)Chevalley condition\(^1\)

3. there is a generic $\tau \in \mathcal{P}_F(\Sigma)$ such that every $\phi \in \mathcal{P}_F(I)$ is isomorphic to $p^* \tau$ for some $p : I \to \Sigma$.

\(^1\)i.e. we have $v^* \exists_u \vdash \exists_p q^*$ and $v^* \forall_u \vdash \forall_p q^*$ for all pullbacks

\[\begin{array}{ccc}
L & \xrightarrow{q} & J \\
\downarrow p & & \downarrow u \\
K & \xrightarrow[v]{} & I
\end{array}\]
If $F$ is just a tripos then the third condition has to be weakened as follows:

for very $I \in S$ there is a $P(I)$ in $S$ and $\varepsilon_I$ in $\mathcal{P}_F(I \times P(I))$ such that for every $\rho$ in $\mathcal{P}_F(I \times J)$

$$(\text{Comp}) \quad \forall j \in J. \exists p \in P(I). \forall i \in I. \rho(i, j) \leftrightarrow i \varepsilon_I p$$

holds in the logic of $\mathcal{P}_F$.

This is the usual *comprehension principle* for HOL. Its Skolemized (and thus stronger) version is equivalent to the existence of a generic subterminal $\tau : T \rightarrow F\Sigma$ (where $\Sigma$ is $P(1)$).

But the logic of the tripos does not validate extensionality for predicates, i.e. $p$ is not uniquely determined by $j$. 

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For triposes $F : S \to \mathcal{E}$ the CO functor $S \to S[\mathcal{P}_F]$ is equivalent to $F$ and a tripos $\mathcal{P}$ is equivalent to $\mathcal{P}_F$ where $F$ is the CO functor $S \to S[\mathcal{P}]$ as shown in Pitts’s Thesis.

Here $S[\mathcal{P}]$ is obtained from $\mathcal{P}$ by “adding quotients” defining morphisms as functional relations. The CO functor $S \to S[\mathcal{P}]$ sends $I$ to $(I, eq_I)$ where $eq_I = \exists \delta_I \top_I$. 
Are triposes $F_1, F_2 : S \to \mathcal{E}$ necessarily equivalent?

The answer is in general NO if $S$ is not equal to $\textbf{Set}$ since for sober (e.g. Hausdorff spaces) $X$ and $Y$ there are as many localic geometric morphism $\text{Sh}(Y) \to \text{Sh}(X)$ as there are continuous maps from $Y$ to $X$.

For all natural numbers $n > 0$ the functor

$$F_n : \textbf{Set} \to \textbf{Set} : l \mapsto l^n$$

is a tripos. But $F_n$ and $F_m$ are equivalent iff $n = m$.

Alas, the question is open for traditional triposes over $\textbf{Set}$ since in the above counterexample only $F_1$ is a traditional tripos.
Already in [HJP80] where triposes were introduced it was asked whether localic toposes $\text{Sh}(A)$ over $\textbf{Set}$ may be induced by traditional triposes whose constant objects functor is not equivalent to $\Delta : \textbf{Set} \to \text{Sh}(A)$.

Maybe we get such examples via classical realizability? Krivine’s criterion (absence of “parallel or”) for a realizability algebra only guarantees that the associated tripos is not localic but not that the induced topos is not localic...e.g. possibly $\textbf{Set}$.

Also realizability toposes $\text{RT}(A)$ over $\textbf{Set}$ could be induced by triposes whose constant objects functor is not equivalent to $\nabla : \textbf{Set} \to \text{RT}(A)$.
If $\mathcal{E}$ is the topos of reflexive graphs $\text{Set}^{\Delta_1^{\text{op}}}$ or the topos $\text{Set}^{\Delta^{\text{op}}}$ of simplicial sets then $\nabla : \text{Set} \to \mathcal{E}$ (right adjoint to $\Gamma = \mathcal{E}(1, -)$) is a (strong) tripos which, however, is not traditional.

Every reflexive graph may be covered by a subobject of some $\nabla(S)$!

Possibly, this also holds for the topos of cubical sets $\text{Set}^{\Box^{\text{op}}}$ (where $\Box$ is the full subcat of $\text{Poset}$ on finite powers of the ordinal 2)?
Together with P. Lietz I showed that the extensional realizability topos $\text{Ext}$ doesn’t validated Ishihara’s $\text{BD}_\mathbb{N}$.

But $\text{Ext}$ validates a negative form of Church’s Thesis, namely

$$\forall f : \mathbb{N} \to \mathbb{N}. \neg \neg \exists e : \mathbb{N}. f = \{e\}$$

and thus is not conservative over $\text{Set}$.

But for every finite limit preserving functor $F : S \to \mathcal{E}$ between toposes the comma category $\mathcal{E} \downarrow F$ is a topos and the functor $P_F = \partial_1 = \text{cod} : \mathcal{E} \downarrow F \to S$ is logical and has full and faithful left and right adjoints sending $I \in S$ to $0 \to FI$ and $\text{id}_{FI}$, respectively.

For triposes $F : \text{Set} \to \mathcal{E}$ the comma category $\mathcal{E} \downarrow F$ is a topos and $P_F = \text{cod} : \mathcal{E} \downarrow F \to \text{Set}$ is logical.

Thus $\mathcal{E} \downarrow F$ only validates sentences which hold in $\text{Set}$ and thus is a neutral model of constructive mathematics.
Summary

- Ground models are typically not unique! (Since $\mathbf{Set}$ is induced by infinitely many non-equivalent triposes over $\mathbf{Set}$).
- Question open for traditional triposes over $\mathbf{Set}$ even for localic and realizability toposes though there are canonical candidates $\Delta$ and $\nabla$, respectively. But are these the only possibilities?
- Triposes $F$ over $\mathbf{Set}$ via “Artin Glueing” give rise to neutral models $\mathcal{E} \uparrow F$ since $P_F = \text{cod} : \mathcal{E} \uparrow F \to \mathbf{Set}$ is logical.
- With a bit of luck $\mathcal{E} \uparrow F$ preserves some of the weaknesses of $\mathcal{E}$, e.g. doesn’t validate $\text{FAN}$, $\text{BD}_\mathbb{N}$, etc.
A. Miquel has introduced a notion of *implicative algebra* and shown that every such i.a. $A$ gives rise to a tripos $\mathcal{P}^A$ over $\textbf{Set}$ and every traditional tripos over $\textbf{Set}$ arises this way as $\Delta_A : \textbf{Set} \to \textbf{Set}[\mathcal{P}^A]$.

This generalizes to base toposes $S$ with nno: every traditional strong tripos $F : S \to \mathcal{E}$ is equivalent to $\Delta_A : S \to S[\mathcal{P}^A]$ for some i.a. $A$ in $S$ where $\Delta_A(I) = (I, eq_I)$. 

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An implicative structure is a complete lattice $\mathcal{A} = (A, \leq)$ together with an implication operation $\rightarrow: \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{A}$ such that $y \rightarrow \bigwedge X = \bigwedge \{y \rightarrow x\}$ for all $y \in \mathcal{A}$ and $X \subseteq \mathcal{A}$.

Thus $y \rightarrow (-)$ has a left adjoint $(-)y$ given by

$$xy = \bigwedge\{z \mid x \leq y \rightarrow z\}$$

Then $K_A = \bigwedge_{x, y \in \mathcal{A}} x \rightarrow y \rightarrow x$ and $S_A = \bigwedge_{x, y, z \in \mathcal{A}} (x \rightarrow y \rightarrow z) \rightarrow (x \rightarrow y) \rightarrow x \rightarrow z$ are elements of $\mathcal{A}$ for which we have

$$K_A xy \leq x \quad \text{and} \quad S_A xyz = xz(yz)$$
A separator in an implicative structure \((\mathcal{A}, \rightarrow)\) is an upward closed subset \(S\) of \(\mathcal{A}\) such that \(K_\mathcal{A}, S_\mathcal{A} \in S\) and \(S\) is closed under *modus ponens*, i.e. \(b \in S\) whenever \(a \in S\) and \(a \rightarrow b \in S\).

An implicative algebra is a triple \((\mathcal{A}, \rightarrow, S)\) such that \((\mathcal{A}, \rightarrow)\) is an implicative structure and \(S\) is a separator in \((\mathcal{A}, \rightarrow)\).

With every implicative algebra \(\mathcal{A}\) one associates a *Set*-based tripos \(\mathcal{P}^\mathcal{A}\) where \(\mathcal{P}^\mathcal{A}(I)\) is the preorder \(\vdash_I\) on \(\mathcal{A}^I\) defined as

\[
\varphi \vdash_I \psi \quad \text{iff} \quad \bigwedge_{i \in I} (\varphi_i \rightarrow \psi_i) \in S
\]

and reindexing is given by precomposition.