Counterexamples in Cubical Sets

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Definition

*Brouwer’s principle* states that all functions \( \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N} \) are continuous.

Theorem (S.)

*Working in a metatheory where Brouwer’s principle holds, weak forms of countable choice and collection are false in cubical sets.*
We work over intensional type theory.

**Definition**

A type $X$ is an *hproposition* if the type $\prod_{x,y:X} x = y$ is inhabited.

A type $X$ is an *hset* if for all $x, y : X$, the type $x = y$ is an hproposition.

**Definition**

Given a type $X$, we define the *propositional truncation of $X$*, $\|X\|$, to be the higher inductive type defined as follows.

1. For any element $x$ of $X$ there is an element $|x|$ of $\|X\|$.
2. For any two elements $x, y$ of $\|X\|$ there is an equality $x = y$. 
Definition

The *axiom of choice* states that for every hset $X$ and every $Y : X \rightarrow hSet$, we have the following

$$\prod_{x : X} \parallel Y(x) \parallel \rightarrow \parallel \prod_{x : X} Y(x) \parallel$$
Definition
The axiom of choice states that for every hset $X$ and every $Y : X \to \text{hSet}$, we have the following

$$\prod_{x:X} \| Y(x) \| \rightarrow \| \prod_{x:X} Y(x) \|$$

We usually work with restricted versions of the full axiom, e.g.

Definition
Write $\text{AC}_{\mathbb{N},2}$ for the following choice axiom. Suppose we are given $P, Q : \mathbb{N} \to \text{hProp}$. Then,

$$\prod_{n:\mathbb{N}} \| P(n) + Q(n) \| \rightarrow \| \prod_{n:\mathbb{N}} P(n) + Q(n) \|$$
Definition (Bridges, Richman, Schuster)

We refer to the following choice axiom as \textit{weak countable choice}. For all \( X : \mathbb{N} \rightarrow \text{hSet} \) such that

\[
\prod_{m \neq n} \| \text{isContr}(X(m)) + \text{isContr}(X(n)) \|
\]

we have

\[
\prod_{n : \mathbb{N}} \|X(n)\| \rightarrow \left\| \prod_{n : \mathbb{N}} X(n) \right\|
\]

Note that \( \text{AC}_{\mathbb{N}, 2} \) and weak countable choice follow from the law of excluded middle.
Definition
Given $\alpha : \mathbb{N} \rightarrow 2$, write $\langle \alpha \rangle$ for the type
$\sum_{n : \mathbb{N}} \alpha(n) = 1 \times \prod_{m < n} \alpha(m) = 0$. “There is a (necessarily
unique) least $n$ such that $\alpha(n) = 1$.”

Definition (Escardó-Knapp)
We call the following axiom Escardó-Knapp choice, $\textbf{EKC}$. For
every hset $X$, and every binary sequence $\alpha : \mathbb{N} \rightarrow 2$,

$$(\langle \alpha \rangle \rightarrow \|X\|) \rightarrow \|\langle \alpha \rangle \rightarrow X\|$$
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I also consider the “intersection” of EKC and $\text{AC}_{\mathbb{N},2}$.

Definition
We refer to EKC2 as the axiom that for any $P, Q : \text{hProp}$, we have

$$(\langle \alpha \rangle \to \|P + Q\|) \rightarrow \|\langle \alpha \rangle \to P + Q\|$$
Definition (Cohen, Coquand, Huber, Mörtberg)

The *cube category* is the category where \( \mathbb{N} \) is the set of objects and a morphism from \( m \) to \( n \) is a homomorphism from the free De Morgan algebra on \( m \) elements to the free De Morgan algebra on \( n \) elements. A *cubical set* is a functor from the cube category to sets.

Theorem (Cohen, Coquand, Huber, Mörtberg)

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Theorem (Cohen, Coquand, Huber, Mörtberg)

*Cubical sets form a constructive model of homotopy type theory.*
We think of a cubical set $X$ as a topological space. We think of elements of $X(0)$ as “points”, elements of $X(1)$ as “paths” and elements of $X(2)$ as “homotopies between paths.”

We have a diagram

$$
\begin{array}{c}
X(1) \\
\delta_0
\end{array} 
\xrightarrow{i} 
\begin{array}{c}
X(0) \\
\delta_1
\end{array}
$$

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$$
\begin{array}{ccc}
X(1) & \xleftarrow{i} & X(0) \\
\delta_0 & \cong & \delta_1 \\
\end{array}
$$

We refer to paths in the image of $i$ as \textit{constant} or \textit{degenerate}.

Note that even for hsets elements of $X(2)$ play a non trivial role: Any two paths with the same endpoints are homotopic, but sometimes we can also show strict equality (equal as elements of the set $X(1)$).
Propositional truncation exists in cubical sets. It has rich structure, in contrast to propositional truncation in models of extensional type theory.

∥X∥ contains a subobject LFR(∥X∥) (local fibrant replacement) such that

1. LFR(∥X∥) is a locally decidable, i.e. every element of ∥X∥ either belongs to LFR(∥X∥) or does not. In particular every path in ∥X∥ belongs to LFR or does not.

2. Every point of ∥X∥ (and hence every constant path) belongs to LFR(∥X∥).

3. LFR(∥X∥) is equivalent to ∥X∥.

We will refer to the elements of ∥X∥ belonging to LFR(∥X∥) as squash free.
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\( \| X \| \) contains a subobject \( \text{LFR}(X) \) (local fibrant replacement) such that

1. \( \text{LFR}(X) \) is a *locally decidable* i.e. every element of \( \| X \| \) either belongs to \( \text{LFR}(X) \) or does not. In particular every path in \( \| X \| \) belongs to \( \text{LFR} \) or does not.

2. Every point of \( \| X \| \) (and hence every constant path) belongs to \( \text{LFR}(X) \).

3. \( \text{LFR}(X) \) is equivalent to \( X \).

We will refer to the elements of \( \| X \| \) belonging to \( \text{LFR}(X) \) as *squash free*. 
Theorem

The following are false in cubical sets, assuming Brouwer’s principle.

1. $\prod_N S^1$ is covered by an hset $0\text{-Cov}(\prod_N S^1)$.
2. An Escardó-Knapp variant of fullness, $\text{Full}(\mathbb{N}, 2)_{EK}$
3. An Escardó-Knapp variant of collection, $\text{Coll}_{EK}$
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Main idea of proof: Let $p$ be a path in $\|X\|$, say that $p$ is non degenerate. Write $p_\alpha$ for the path in $\langle \alpha \rangle \to \|X\|$ constantly equal to $p$. Note that $p_\alpha$ is degenerate if and only if $\alpha = 0^\omega$. 
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Any natural transformation $f : \langle \alpha \rangle \to \|X\| \to \|\langle \alpha \rangle \to X\|$ restricts to a function $f_1$ from paths in $\langle \alpha \rangle \to \|X\|$ to paths in $\|\langle \alpha \rangle \to X\|$ that preserves degenerate maps.
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Since $p_0^\omega$ is degenerate, $f_1(p_0^\omega)$ is squash free.
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Since $p_{0^\omega}$ is degenerate, $f_1(p_{0^\omega})$ is squash free.

Hence by continuity there is a natural number $n$ such that $f_1(p_n)$ is squash free. We thus obtain a path in $X$. 

Corollary

The following are false in cubical sets, assuming Brouwer’s principle. They are independent of homotopy type theory.

1. PAx
2. Dependent choice, DC
3. WISC
4. Fullness, Full
5. Collection, Coll
6. $\prod_N S^1$ is connected, $\prod_N S^1$-Conn
7. (Bridges-Richman-Schuster) Weak countable choice, WCC
8. $AC_{N,2}$
9. Escardó-Knapp choice, EKC

Proof.
See next slide.
Corollary

Work over $\text{CZF}_{\text{Exp,Rep}}$, the theory obtained by replacing subset collection with exponentiation and strong collection with replacement in $\text{CZF}$. The following are not provable.

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2. Dependent choice, DC
3. WISC
4. Fullness, Full
5. Collection, Coll
6. (Bridges-Richman-Schuster) Weak countable choice, WCC
7. $\text{AC}_{\mathbb{N},2}$
8. Escardó-Knapp choice, EKC

Proof.
The HIT cumulative hierarchy models $\text{CZF}_{\text{Exp,Rep}}$ and the principles $\text{Coll}_{\text{EK}}$ and $\text{Full}(\mathbb{N},2)_{\text{EK}}$ are both “absolute” for the HIT cumulative hierarchy.
Further questions:

1. Is there a constructive model of homotopy type theory with countable choice?

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Thank you for your attention!