Counterexamples in Cubical Sets

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Definition

Brouwer's principle states that all functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous.

Theorem (S.)

Working in a metatheory where Brouwer's principle holds, weak forms of countable choice and collection are false in cubical sets.

We work over intensional type theory.

Definition

A type X is an *hproposition* if the type $\prod_{x,y:X} x = y$ is inhabited.

A type X is an *hset* if for all $x, y : X$, the type $x = y$ is an hproposition.

Definition

Given a type X, we define the *propositional truncation of* X, $\|X\|$ to be the higher inductive type defined as follows.

1. For any element x of X there is an element $|x|$ of $\|X\|$.
2. For any two elements x, y of $\|X\|$ there is an equality $x = y$.

Definition

The *axiom of choice* states that for every hset X and every $Y : X \to \mathbf{hSet}$, we have the following

$$\prod_{x:X} \|Y(x)\| \longrightarrow \left\| \prod_{x:X} Y(x) \right\|$$

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We usually work with restricted versions of the full axiom, e.g.

Definition

Write $\mathbf{AC}_{\mathbb{N},2}$ for the following choice axiom. Suppose we are given $P, Q : \mathbb{N} \to \mathbf{hProp}$. Then,

$$\prod_{n:\mathbb{N}} \|P(n) + Q(n)\| \longrightarrow \left\|\prod_{n:\mathbb{N}} P(n) + Q(n)\right\|$$

Definition (Bridges, Richman, Schuster)

We refer to the following choice axiom as *weak countable choice*. For all $X : \mathbb{N} \to \mathbf{hSet}$ such that

$$\prod_{m \neq n} \|\mathbf{isContr}(X(m)) + \mathbf{isContr}(X(n))\|$$

we have

$$\prod_{n:\mathbb{N}} \|X(n)\| \longrightarrow \left\| \prod_{n:\mathbb{N}} X(n) \right\|$$

Note that $\mathbf{AC}_{\mathbb{N},2}$ and weak countable choice follow from the law of excluded middle.

Definition

Given $\alpha \colon \mathbb{N} \to 2$, write $\langle \alpha \rangle$ for the type $\sum_{n:\mathbb{N}} \alpha(n) = 1 \times \prod_{m<n} \alpha(m) = 0$. "There is a (necessarily unique) least n such that $\alpha(n) = 1$."

Definition (Escardó-Knapp)

We call the following axiom *Escardó-Knapp choice*, **EKC**. For every hset X, and every binary sequence $\alpha \colon \mathbb{N} \to 2$,

$$(\langle \alpha \rangle \to \|X\|) \longrightarrow \|\langle \alpha \rangle \to X\|$$

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I also consider the "intersection" of **EKC** and $\mathbf{AC}_{\mathbb{N},2}$.

Definition

We refer to **EKC**2 as the axiom that for any P, Q : **hProp**, we have

$$(\langle \alpha \rangle \to \|P + Q\|) \longrightarrow \|\langle \alpha \rangle \to P + Q\|$$

Definition (Cohen, Coquand, Huber, Mörtberg)

The *cube category* is the category where $\mathbb{N}$ is the set of objects and a morphism from m to n is a homomorphism from the free De Morgan algebra on m elements to the free De Morgan algebra on n elements. A *cubical set* is a functor from the cube category to sets.

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Theorem (Cohen, Coquand, Huber, Mörtberg)

Cubical sets form a constructive model of homotopy type theory.

We think of a cubical set X as a topological space. We think of elements of $X(0)$ as “points”, elements of $X(1)$ as “paths” and elements of $X(2)$ as “homotopies between paths.”

We have a diagram

$$X(1) \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} X(0), \qquad X(1) \xleftarrow{\;i\;} X(0)$$

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Note that even for hsets elements of $X(2)$ play a non trivial role: Any two paths with the same endpoints are homotopic, but sometimes we can also show strict equality (equal as elements of the set $X(1)$).

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$\|X\|$ contains a subobject $\mathrm{LFR}(X)$ (local fibrant replacement) such that

1. $\mathrm{LFR}(X)$ is a *locally decidable* i.e. every element of $\|X\|$ either belongs to $\mathrm{LFR}(X)$ or does not. In particular every path in $\|X\|$ belongs to LFR or does not.
2. Every point of $\|X\|$ (and hence every constant path) belongs to $\mathrm{LFR}(X)$.
3. $\mathrm{LFR}(X)$ is equivalent to X.

We will refer to the elements of $\|X\|$ belonging to $\mathrm{LFR}(X)$ as *squash free*.

Theorem

The following are false in cubical sets, assuming Brouwer's principle.

1. $\prod_{\mathbb{N}} \mathbb{S}^1$ *is covered by an hset* $0\text{-}\mathbf{Cov}(\prod_{\mathbb{N}} \mathbb{S}^1)$.
2. *An Escardó-Knapp variant of fullness,* $\mathbf{Full}(\mathbb{N}, 2)_{\mathbf{EK}}$
3. *An Escardó-Knapp variant of collection,* $\mathbf{Coll}_{\mathbf{EK}}$

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Main idea of proof: Let p be a path in $\|X\|$, say that p is non degenerate. Write p_α for the path in $\langle \alpha \rangle \to \|X\|$ constantly equal to p. Note that p_α is degenerate if and only if $\alpha = 0^\omega$.

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Any natural transformation $f : \langle\alpha\rangle \to \|X\| \longrightarrow \|\langle\alpha\rangle \to X\|$ restricts to a function f_1 from paths in $\langle\alpha\rangle \to \|X\|$ to paths in $\|\langle\alpha\rangle \to X\|$ that preserves degenerate maps.

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Since p_{0^ω} is degenerate, $f_1(p_{0^\omega})$ is squash free.
Hence by continuity there is a natural number n such that $f_1(p_{\underline{n}})$ is squash free. We thus obtain a path in X.

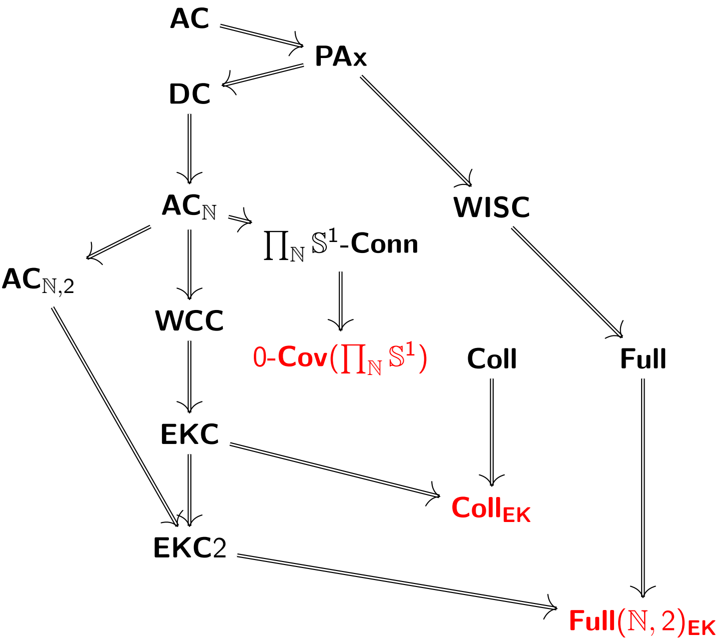
Corollary

The following are false in cubical sets, assuming Brouwer's principle. They are independent of homotopy type theory.

1. **PAx**
2. *Dependent choice,* **DC**
3. **WISC**
4. *Fullness,* **Full**
5. *Collection,* **Coll**
6. $\prod_{\mathbb{N}} \mathbb{S}^1$ *is connected,* $\prod_{\mathbb{N}} \mathbb{S}^1$**-Conn**
7. *(Bridges-Richman-Schuster) Weak countable choice,* **WCC**
8. $\mathbf{AC}_{\mathbb{N},2}$
9. *Escardó-Knapp choice,* **EKC**

Proof.

See next slide. □



AC
PAx
DC
$AC_{\mathbb{N}}$
WISC
$\prod_{\mathbb{N}} \mathbb{S}^1$-Conn
$AC_{\mathbb{N},2}$
WCC
0-Cov($\prod_{\mathbb{N}} \mathbb{S}^1$)
Coll
Full
EKC
$Coll_{EK}$
EKC2
Full$(\mathbb{N}, 2)_{EK}$

Corollary

Work over $\mathbf{CZF_{Exp,Rep}}$*, the theory obtained by replacing subset collection with exponentiation and strong collection with replacement in* **CZF**. *The following are not provable.*

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2. *Dependent choice,* **DC**
3. **WISC**
4. *Fullness,* **Full**
5. *Collection,* **Coll**
6. *(Bridges-Richman-Schuster) Weak countable choice,* **WCC**
7. $\mathbf{AC}_{\mathbb{N},2}$
8. *Escardó-Knapp choice,* **EKC**

Proof.

The HIT cumulative hierarchy models $\mathbf{CZF_{Exp,Rep}}$ and the principles $\mathbf{Coll_{EK}}$ and $\mathbf{Full}(\mathbb{N}, 2)_{\mathbf{EK}}$ are both "absolute" for the HIT cumulative hierarchy. □

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1. Is there a constructive model of homotopy type theory with countable choice?
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Thank you for your attention!