Families of Sets in Constructive Measure Theory

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MloC 19

Stockolm, 22.09.2019

Outline

1. Motivation
2. Partial functions and complemented subsets in Bishop's set theory
3. Set-indexed families of partial functions and complemented subsets
4. Impredicativities in Bishop-Cheng measure theory
5. Pre-measure and pre-integration spaces

Historical developements

- Bishop was not particularly satisfied with the generality of the measure theory (**BMT**) developed in [Bishop, 1967]
- Bishop-Cheng measure theory (**BCMT**) is developed in [Bishop and Cheng, 1972] and extended in chapter 6 of [Bishop and Bridges, 1985]

Recent developments

- Pointfree, algebraic approach to constructive measure theory in [Coquand and Palmgren, 2002] and [Spitters, 2005], [Spitters, 2006] to avoid impredicativities.
- Recent work: Formalization in Coq, see [Semeria, 2019]. A metric approach in [Ishihara, 2017] and constructive probability theory in [Chan, 2019]

Goal

- Work within **BISH**
- Using tools from Bishop's set theory, i.e. set-indexed families
- Towards a predicative formulation of **BCMT**

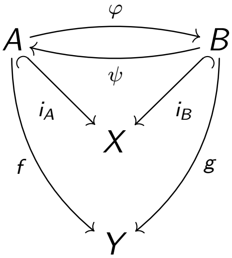
A partial function from X to Y is a triple (A, i_A, f) where (A, i_A) is a subset of X and $f : A \to Y$ is a function, we write $f : X \rightharpoonup Y$.

The totality $\mathbb{F}^{\rightharpoonup}(X, Y)$ of partial functions is not a set as this would imply that $\mathcal{P}(X)$ would be a set as well. We write

$$\mathcal{F}(X) := \mathbb{F}^{\rightharpoonup}(X, \mathbb{R})$$

for the totality of real-valued partial functions.

Two partial functions $(A, i_A, f), (B, i_B, g)$ are equal if there are functions $\varphi : A \to B$ and $\psi : B \to A$ s.t. the following diagrams commute



In this case we write $(\varphi, \psi) : (A, i_A, f) =_{\mathbb{F}^{\rightharpoonup}(X,Y)} (B, i_B, g)$.

Let X be a set with an inequality $\neq_X$, a complemented subset of X is a quadruple (A, i_A, B, i_B) where (A, i_A) and (B, i_B) are subsets of X s.t.

$$\forall a \in A \, \forall b \in B : \; i_A(a) \neq_X i_B(b)$$

For any complemented subset $\boldsymbol{A} = (A^1, A^0)$ the characteristic function $\chi_{\boldsymbol{A}} : A^1 \cup A^0 \to \mathbf{2}$ is defined as

$$\chi_{\boldsymbol{A}}(x) := \begin{cases} 1, & \text{if } x \in A^1 \\ 0, & \text{if } x \in A^0 \end{cases}$$

For $\boldsymbol{A} = (A^1, A^0)$ and $\boldsymbol{B} = (B^1, B^0)$ we have operations

- $\boldsymbol{A} \wedge \boldsymbol{B} := \big(A^1 \cap B^1,\ (A^1 \cap B^0) \cup (A^0 \cap B^1) \cup (A^0 \cap B^0)\big)$
- $\boldsymbol{A} \vee \boldsymbol{B} := \big((A^1 \cap B^0) \cup (A^0 \cap B^1) \cup (A^1 \cap B^1),\ A^0 \cap B^0\big)$
- $-\boldsymbol{A} := (A^0, A^1)$ Note that $- - \boldsymbol{A} = \boldsymbol{A}$

Two complementes subsets $\boldsymbol{A} = (A^1, A^0)$ and $\boldsymbol{B} = (B^1, B^0)$ are equal if

$$\boldsymbol{A} =_{\mathcal{P}^{][}(X)} \boldsymbol{B} \ :\Leftrightarrow\ A^1 =_{\mathcal{P}(X)} B^1 \ \&\ A^0 =_{\mathcal{P}(X)} B^0$$

Again, the totality $\mathcal{P}^{][}(X)$ of complemented subsets of X is not a set.

Families of complemented subsets

Let X have a fixed apartness relation $\neq_X$, a family of complemented subsets of X indexed by I is a sextuple

$$\boldsymbol{\lambda} = (\lambda_0^1, \mathcal{E}^1, \lambda_1^1, \lambda_0^0, \mathcal{E}^0, \lambda_1^0)$$

where $\lambda^1 = (\lambda_0^1, \mathcal{E}^1, \lambda_1^1)$ and $\lambda^0 = (\lambda_0^0, \mathcal{E}^0, \lambda_1^0)$ are I-families of subsets s.t.

$$\forall i \in I \; \forall x \in \lambda_0^1(i) \; \forall y \in \lambda_0^0(i) : \; \varepsilon_i^1(x) \neq_X \varepsilon_i^0(y)$$

i.e. for all $i \in I$ we have a complemented subset

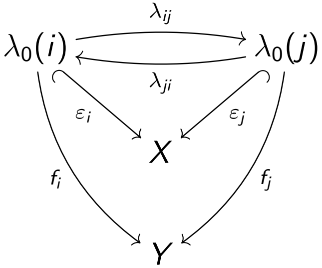
$$\boldsymbol{\lambda}_0(i) := \big(\lambda_0^1(i), \lambda_0^0(i)\big)$$

Families of partial functions

A family of partial functions from X to Y indexed by I is a quadruple $\mathbf{\Lambda} = (\lambda_0, \mathcal{E}, \lambda_1, F)$, where

- $\lambda_{\mathbf{\Lambda}} = (\lambda_0, \mathcal{E}, \lambda_1)$ is an I-family of subsets of X
- $F : \bigcurlywedge_{i \in I} \mathbb{F}(\lambda_0(i), Y)$ where $f_i := F(i)$

s.t. for $i =_I j$ the following diagrams commute



Impredicativities in Bishop-Cheng measure theory

1. A measure space contains a set of complemented subsets, an integration space contains a set of partial functions.
2. The definition of a measure space contains quantifiers over all complemented subsets, thus presupposing that $\mathcal{P}^{][}(X)$ is a set.
3. The definition of the complete extension of an integration space takes the totality of integrable function L_1 to be a set, thus presupposing that $\mathcal{F}(X)$ is a set.

Avoiding impredicativities I

An I-family $\boldsymbol{\lambda} = (\lambda_0^1, \mathcal{E}^1, \lambda_1^1, \lambda_0^0, \mathcal{E}^0, \lambda_1^0)$ of complemented subsets is called an I-set of complemented subsets if

$$\forall i, j \in I : \ \boldsymbol{\lambda}_0(i) =_{\mathcal{P}^{][}(X)} \boldsymbol{\lambda}_0(j) \ \Leftrightarrow \ i =_I j$$

A measure space is thus actually a quadruple $(X, I, \boldsymbol{\lambda}, \mu)$ where the index set is implicitly given.

An I-family $\boldsymbol{\Lambda} = (\lambda_0, \mathcal{E}, \lambda_1, F)$ of partial functions is called an I-set of partial functions if

$$\forall i, j \in I : \ f_i =_{\mathcal{F}(X)} f_j \ \Leftrightarrow \ i =_I j$$

An integration space is thus actually a quadruple $(X, I, \boldsymbol{\Lambda}, \int)$ where the index set is implicitly given.

Avoiding impredicativities II

In [Bishop, 1967, p.183] problem 2 is avoided:

"Let $\mathfrak{F}$ be any family of complemented subsets of X [...] Let $\mathfrak{M}$ be a subfamily of $\mathfrak{F}$ closed under finite unions, intersections, and differences. Let the function $\mu : \mathfrak{M} \to \mathbb{R}^{0+}$ satisfy the following conditions [...]"

A measure space is of the form $(X, I, \boldsymbol{\lambda}, J, \boldsymbol{\nu}, \mu)$, where $\boldsymbol{\lambda}$ is an I-family and $\boldsymbol{\nu}$ is a J-family of complemented subsets s.t. $\boldsymbol{\lambda}$ is a subfamily of $\boldsymbol{\nu}$.

Quantification over $\mathcal{P}^{][}(X)$ is replaced by quantification over J.

Bishop's proposal

on formalization in "Mathematics as a numerical language"

"A *measure space* is a family $\mathcal{M} \equiv \{A_t\}_{t \in T}$ of complemented subsets of a set X [...], a map $\mu : T \to \mathbb{R}^{0+}$ and an additional structure as follows: [...] If t and s are in T, there exists an element $s \vee t$ of T such that $A_{s \vee t} < A_t \cup A_s$. Similarly, there exist operations $\wedge$ and $\sim$ on T, corresponding to the set-theoretic operations $\cap$ and $-$."

- [Bishop, 1970, p. 67]

Pre-measure space

Let X be a set with an apartness-relation $\neq_X$, I, J sets,

- $\boldsymbol{\lambda} = (\lambda_0^1, \lambda_1^1, \mathcal{E}^1, \lambda_0^0, \lambda_1^0, \mathcal{E}^0)$ an I-set
- $\boldsymbol{\nu} = (\nu_0^1, \nu_1^1, E^1, \nu_0^0, \nu_1^0, E^0)$ a J-set of complemented subsets of X

s.t. $\boldsymbol{\lambda}$ is a subfamily of $\boldsymbol{\nu}$ (i.e. we have an embedding $h : I \hookrightarrow J$) and $\mu : I \to \mathbb{R}_{\geq 0}$ a function.

Furthermore, assume that we have assignment routines $\wedge : J \times J \rightsquigarrow J$, $\vee : J \times J \rightsquigarrow J$ and $\sim : J \rightsquigarrow J$, as well as $\wedge : I \times I \rightsquigarrow I$, $\vee : I \times I \rightsquigarrow I$ and $\sim : I \times I \rightsquigarrow I$ s.t. for all $i, j \in I$ we have

- $h(i \wedge j) =_J h(i) \wedge h(j)$
- $h(i \vee j) =_J h(i) \vee h(j)$
- $h(i \sim j) =_J h(i) \wedge \sim h(j)$

Then $(X, I, \boldsymbol{\lambda}, J, \boldsymbol{\nu}, \mu)$ is a pre-measure space if the following conditions hold:

1. $\forall i, j \in J$ we have
   - $\boldsymbol{\nu}_0(i \wedge j) =_{\mathcal{P}]\![(X)} \boldsymbol{\nu}_0(i) \wedge \boldsymbol{\nu}_0(j)$
   - $\boldsymbol{\nu}_0(i \vee j) =_{\mathcal{P}]\![(X)} \boldsymbol{\nu}_0(i) \vee \boldsymbol{\nu}_0(j)$
   - $\boldsymbol{\nu}_0(\sim i) =_{\mathcal{P}]\![(X)} -\boldsymbol{\nu}_0(i)$

   and for $i, j \in I$ we have that
   $\mu(i) + (j) =_{\mathbb{R}} \mu(i \vee j) + \mu(i \wedge j)$.
2. $\forall i \in I\ \forall j \in J$: If there is a $k \in I$ s.t. $h(k) =_J h(i) \wedge j$, then there exist $l \in I$ s.t. $h(l) =_J h(i) \wedge \sim j$ and $\mu(i) =_{\mathbb{R}} \mu(k) + \mu(l)$.
3. $\exists i \in I$ s.t. $\mu(i) > 0$.
4. $\forall \alpha \in \mathbb{F}(\mathbb{N}, I)$: If $\ell := \lim_{m \to \infty} \mu(\bigwedge_{n=1}^{m} \alpha_n)$ exists and $\ell > 0$, then there is a $x \in \bigcap_{n \in \mathbb{N}} \lambda_0^1(\alpha_n)$ (i.e. $\bigcap_{n \in \mathbb{N}} \lambda_0^1(\alpha_n)$ is inhabited).

Pre-integration space (of partial functions)

Let X be a set, I a set, $\mathbf{\Lambda} = (\lambda_0, \lambda_1, \mathcal{E}, F)$ an I-set of real-valued partial functions and $\int : I \to \mathbb{R}$ a function. Furthermore, assume that we have assignment routines

$$
\begin{aligned}
\_ \cdot \_ &: \mathbb{R} \times I \rightsquigarrow I \\
\_ + \_ &: I \times I \rightsquigarrow I \\
|\_| &: I \rightsquigarrow I \\
\wedge_1 &: I \rightsquigarrow I
\end{aligned}
$$

Then $(X, I, \mathbf{\Lambda}, \int)$ is called a pre-integration space if the following conditions hold

1. $\forall i, j \in I \; \forall a, b \in \mathbb{R}$ we have
   - $f_{a \cdot i + b \cdot j} =_{\mathcal{F}(X)} a f_i + b f_j$
   - $f_{|i|} =_{\mathcal{F}(X)} |f_i|$
   - $f_{\wedge_1(i)} =_{\mathcal{F}(X)} f_i \wedge 1$

   and we have that $\int (a \cdot i + b \cdot j) =_{\mathbb{R}} a \int i + b \int j$
2. $\forall i \in I \; \forall \alpha \in \mathbb{F}(\mathbb{N}, I)$ s.t.
   - $\forall m \in \mathbb{N} : \; f_{\alpha_m} \geq 0$
   - $\ell := \sum_{k=1}^{\infty} \int \alpha_k$ exists and $\ell < \int i$

   there is $x \in \lambda_0(i) \cap \left(\bigcap_{n \in \mathbb{N}} \lambda_0(\alpha_n) \right)$ s.t. $\ell' := \sum_{k=1}^{\infty} f_{\alpha_k}(x)$ exists and $\ell' < f_i(x)$.
3. $\exists i \in I$ s.t. $\int i =_{\mathbb{R}} 1$
4. $\forall i \in I \; \forall \alpha, \beta \in \mathbb{F}(\mathbb{N}, I)$ s.t. $\alpha_m =_I m \cdot (\wedge_1(m^{-1} \cdot i))$ and $\beta_m =_I m^{-1} \cdot (\wedge_1(m \cdot |i|))$ for all $m \in \mathbb{N}$, we have that $\ell := \lim_{n \to \infty} \int \alpha_n$ and $\ell' := \lim_{n \to \infty} \int \beta_n$ exist and $\ell =_{\mathbb{R}} \int i$ and $\ell' =_{\mathbb{R}} 0$.

Working with pre-integration spaces and pre-measure spaces

What we can do so far

- Give concrete examples of pre-measure spaces (set of detachable subsets with Dirac measure)
- Construct the pre-integration space of simple functions over a pre-measure space
- Construct a predicative version of the complete extension of a pre-integration space.

1-Norm

Let $(X, I, \mathbf{\Lambda}, \int)$ be a pre-integration space

$$i =_{\int} j \;:\Leftrightarrow\; \int |i - j| =_{\mathbb{R}} 0$$

defines an equality on I and $(I, =_{\int})$ is a $\mathbb{R}$-vector space. Moreover the assignment routine $\|\_\|_1 : I \rightsquigarrow \mathbb{R}_{\geq 0}$ with $\|i\|_1 := \int |i|$ is a function and defines a norm on $(I, =_{\int})$.

Goal: Find extended pre-integration space $(X, I_1, \mathbf{\Lambda}_1, \int)$ s.t. I_1 is the metric completion w.r.t. the norm $\|\_\|_1$.

Set of representations

$$l_1 := \left\{ \alpha \in \mathbb{F}(\mathbb{N}, I) \;:\; \sum_{n=1}^{\infty} \int |\alpha_n| \text{ exists} \right\}$$

together with the equality

$$\alpha =_{l_1} \beta \;:\Leftrightarrow\; \left(F_\alpha, e_{F_\alpha}, \sum_n f_{\alpha_n}\right) =_{\mathcal{F}(X)} \left(F_\beta, e_{F_\beta}, \sum_n f_{\beta_n}\right)$$

where

$$F_\alpha := \left\{ x \in \bigcap_n \lambda_0(\alpha_n) \;:\; \sum_n |f_{\alpha_n}(x)| \text{ exists} \right\}$$

and F_β is defined accordingly.

Canonically integrable functions

We define the set of canonically integrable functions (see [Spitters, 2002, p. 24]) to be

The I_1-set of partial functions $\mathbf{\Lambda}_1 = (\nu_0, \nu_1, E, G)$ s.t.

- $\nu_0(\alpha) := F_\alpha$
- $G(\alpha) := g_\alpha := \sum_n f_{\alpha_n}$

Through the embedding

$$\begin{aligned} e : I &\hookrightarrow I_1 \\ i &\mapsto (i, 0 \cdot i, 0 \cdot i, ...) \end{aligned}$$

$\mathbf{\Lambda}$ becomes a subfamily of $\mathbf{\Lambda}_1$

Avoiding impredicativities III

The assignment routine $\int : I_1 \rightsquigarrow \mathbb{R}$ with $\int \alpha := \sum_n \int \alpha_n$, is a function that is compatible with the embedding e.

Theorem

$(X, I_1, \mathbf{\Lambda}_1, \int)$ is a pre-integration space and $(I_1, \|\_\|_1)$ is a complete metric space s.t. $(I, \|\_\|_1)$ is a dense subspace via the emedding e.

Note: This is a completely predicative description of the complete extension that doesn't make use of the notion of an integrable function or a full set.

Lebesgue's series theorem (2.15)

Theorem

Let $\Gamma \in \mathbb{F}(\mathbb{N}, l_1)$ *s.t.* $\sum_n \int |\Gamma_n|$ *exists. Then there is a* $\alpha \in l_1$ *s.t.*

$$\nu_0(\alpha) \subseteq \{\, x \in \bigcap_n \nu_0\big(\Gamma_n\big) \;:\; \sum_n |g_{\Gamma_n}(x)| \textit{ exists} \,\}$$

$$\textit{and} \qquad \forall x \in \nu_0(\alpha) :\; g_\alpha(x) = \sum_n g_{\Gamma_n}(x)$$

Furthermore, for any $\alpha \in l_1$ *fulfilling the above condition we have*

$$\lim_{N\to\infty} \int |\alpha - \sum_{n=1}^{N} \Gamma_n| = 0$$

Thank you!

Detachable subsets

Let X be inhabited and define for $\mathbf{2} := \{0,1\}$

$$x \neq_X y \;:\Leftrightarrow\; \exists f \in \mathbb{F}(X,\mathbf{2}) \text{ s.t. } f(x) \neq f(y)$$

Define for $f, g \in \mathbb{F}(X, \mathbf{2})$

- $f \wedge g := fg$
- $f \vee g := f + g - fg$
- $\sim f := 1 - f$

Let $I := J := \mathbb{F}(X, \mathbf{2})$ and let $\boldsymbol{\delta} = (\delta_0^1, \mathcal{E}^1, \delta_1^1, \delta_0^0, \mathcal{E}^0, \delta_1^0)$ be the $\mathbb{F}(X, \mathbf{2})$-family of complemented subset of $(X, \neq_X)$ s.t.

- $\delta_0^1 := \{x \in X : \ f(x) = 1\}$
- $\delta_0^0 := \{x \in X : \ f(x) = 0\}$

Let $x_0 \in X$ and define

$$\begin{aligned} \mu_{x_0} : I &\to \mathbb{R}_{\geq 0} \\ f &\mapsto f(x_0) \end{aligned}$$

Then $(X, I, \boldsymbol{\delta}, J, \boldsymbol{\delta}, \mu_{x_0})$ is a pre-measure space.

Simple functions

Let $(X, I, \boldsymbol{\lambda}, J, \boldsymbol{\nu}, \mu)$ be a pre-measure space. Define

$$S(I) := \sum_{n \in \mathbb{N}} (\mathbb{R} \times I)^n$$

i.e. $S(I)$ is the set of finite sequences of pairs of coefficients in $\mathbb{R}$ and indices in I, together with the equality

$$(a_k, i_k)_{k=1}^n =_{S(I)} (b_\ell, j_\ell)_{\ell=1}^m \;:\Leftrightarrow\; \sum_{k=1}^n a_k \cdot \chi_{\boldsymbol{\lambda}_0(i_k)} =_{\mathcal{F}(X)} \sum_{\ell=1}^m b_\ell \cdot \chi_{\boldsymbol{\lambda}_0(j_\ell)}$$

Let $\mathbf{\Lambda}_{\boldsymbol{\lambda}} = (\lambda_0, \mathcal{E}, \lambda_1, F)$ be the $S(I)$-family of (real-valued) partial functions, s.t. for $v := \sum_{k=1}^{n} a_k \cdot \chi_{\boldsymbol{\lambda}_0(i_k)} \in S(I)$

- $\lambda_0(v) := \bigcap_{k=1}^{n} \left(\lambda_0^1(i_k) \cup \lambda_0^0(i_k)\right)$
- $f_v := \sum_{k=1}^{n} a_k \cdot \chi_{\boldsymbol{\lambda}_0(i_k)}$

Define $\int v \; d\mu := \sum_{k=1}^{n} a_k \cdot \mu(i_k)$

Then $(X, S(I), \mathbf{\Lambda}_{\boldsymbol{\lambda}}, \int \_ \; d\mu)$ is a pre-integration space.

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