

# An algebraic proof of the Frobenius condition

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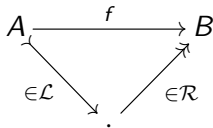
<sup>1</sup>Thanks to the Norwegian Center for Advanced Studies

# Weak factorization systems

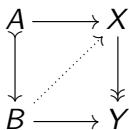
## Definition

A *weak factorization system* on a category  $\mathcal{E}$  consists of two classes of maps  $(\mathcal{L}, \mathcal{R})$ , both closed under retracts, such that:

- (i) Every map  $f : A \longrightarrow B$  factors as a map in  $\mathcal{L}$  followed by one in  $\mathcal{R}$ .



- (ii) In any commutative square with an  $\mathcal{L}$ -map on the left and an  $\mathcal{R}$ -map on the right, there is a diagonal filler making the diagram commute.



# Frobenius

Such structures are part of the definition of a Quillen model category. They are also used to model identity types in HoTT. For that one also needs the following.

## Definition

A weak factorization system  $(\mathcal{L}, \mathcal{R})$  has the *Frobenius property* if the left maps are stable under pullback along the right maps.

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ X & \twoheadrightarrow & Y \end{array}$$

This condition is equivalent to also modelling  $\Pi$ -types.

## Frobenius and $\Pi$

The proof uses the following fact about a wfs  $(\mathcal{L}, \mathcal{R})$  on an LCC  $\mathcal{E}$ .

### Lemma

For any map  $f : Y \rightarrow X$  with base change  $f^* \dashv f_* : \mathcal{E}/Y \rightarrow \mathcal{E}/X$ ,

$$f^* \text{ preserves } \mathcal{L} \quad \text{iff} \quad f_* \text{ preserves } \mathcal{R}.$$

### Proof.

Suppose  $f^*$  preserves  $\mathcal{L}$ , and let  $g : C \rightarrow D$  in  $\mathcal{E}/Y$  be an  $\mathcal{R}$ -map. Test  $f_*g$  against any  $c : A \rightarrow B$  in  $\mathcal{L}$  as on the right below.

$$\begin{array}{ccc} f^*A & \longrightarrow & C \\ \downarrow f^*c & \nearrow \text{dotted} & \downarrow g \\ f^*B & \longrightarrow & B \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & f_*C \\ \downarrow c & \nearrow \text{dotted} & \downarrow f_*g \\ B & \longrightarrow & f_*B \end{array}$$

By adjointness and  $f^*$  preserves  $\mathcal{L}$ , we get a diagonal filler as on the left above; thus there is also one on the right. □

# Frobenius and $\Pi$

## Corollary

Suppose  $\mathcal{E}$  is LCC and the wfs  $(\mathcal{L}, \mathcal{R})$  has the Frobenius property. Then  $\mathcal{R}$  is closed under dependent products ( $\Pi$ -types).

$$\begin{array}{ccc} A & & \Pi_f A \\ \downarrow a & & \downarrow f_* a \\ Y & \xrightarrow{f} & X \end{array}$$

# Quillen model structures

## Definition

A *Quillen model structure* on  $\mathcal{E}$  consists of three classes

$$(\mathcal{C}, \mathcal{W}, \mathcal{F})$$

called cofibrations, weak equivalences, and fibrations, such that:

- (i)  $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$  and  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  are wfs.
- (ii)  $\mathcal{W}$  has the 3-for-2 property.

Let  $\text{TFib} = \mathcal{W} \cap \mathcal{F}$  (“trivial fibrations”),  
and  $\text{TCof} = \mathcal{C} \cap \mathcal{W}$  (“trivial cofibrations”).

# QMS and Frobenius

## Corollary

*Suppose  $\mathcal{E}$  is an LCC with QMS  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ .*

- 1.  $\mathcal{C}$  is stable under all pullbacks iff  
TFib is stable under all pushforwards*
- 2.  $(\text{TCof}, \mathcal{F})$  has Frobenius iff  
 $\mathcal{F}$  has dependent products.*
- 3. If  $\mathcal{C}$  is stable under pullbacks along  $\mathcal{F}$ , then:  
 $(\text{TCof}, \mathcal{F})$  has Frobenius iff  
 $\mathcal{W}$  is stable under pullback along  $\mathcal{F}$  (“right proper”).*

Thus an LCC with a QMS satisfying Frobenius will model MLTT with  $\Sigma, \Pi, \text{Id}$  when taking the fibrations  $\mathcal{F}$  as the types.

# QMS from a premodel

## Definition

A *premodel* in a topos  $\mathcal{E}$  consists of  $(\Phi, \mathbb{I}, \mathbb{V})$  where:

- $\Phi \hookrightarrow \Omega$  is a representable class of monos (satisfying ...).
- $1 \rightrightarrows \mathbb{I}$  is an interval (satisfying ...).
- $\dot{\mathbb{V}} \rightarrow \mathbb{V}$  is a universe (satisfying ...).

Previous work by (A., Coquand, Orton-Pitts, Sattler) provides:

## Construction

*From a premodel  $(\Phi, \mathbb{I}, \mathbb{V})$  one can construct a QMS  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ .*

The **goal for today** is to show:

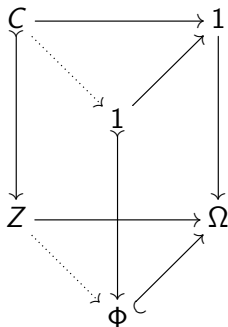
## Theorem

*This model structure has Frobenius (and so is right proper).*



## The cofibration wfs $(\mathcal{C}, \text{TFib})$

The *cofibrations*  $\mathcal{C}$  are the monos  $C \rightarrow Z$  classified by  $\Phi \hookrightarrow \Omega$ .



Note that  $\mathcal{C}$  is therefore stable under all pullbacks.

## The cofibration wfs $(\mathcal{C}, \text{TFib})$

The generic cofibration  $1 \twoheadrightarrow \Phi$  determines a polynomial endofunctor,

$$X^+ := \sum_{\varphi: \Phi} X^{[\varphi]}.$$

This is a (fibered) monad,

$$+ : \mathcal{E} \longrightarrow \mathcal{E},$$

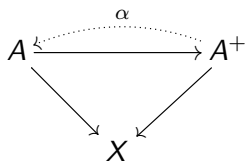
by the *dominance* condition assumed on  $\Phi$ .

The unit  $\eta : X \longrightarrow X^+$  *classifies cofibrant-partial maps* into  $X$ .

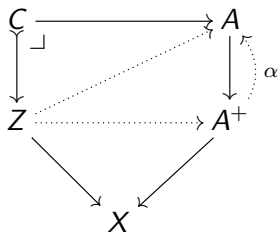
$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \eta \\ Z & \cdots \longrightarrow & X^+ \end{array}$$

## The cofibration wfs $(\mathcal{C}, \text{TFib})$

The *trivial fibrations* are the algebras  $(A, \alpha)$  for the pointed endofunctor  $+_X : \mathcal{E}/X \rightarrow \mathcal{E}/X$ .



Such an algebra  $(A, \alpha)$  has lifts against all cofibrations.



## The cofibration wfs $(\mathcal{C}, \text{TFib})$

The trivial fibrations form the right class of the *cofibration wfs*  $(\mathcal{C}, \text{TFib})$ . The factorization axiom follows from the monad multiplication.

Because  $\mathcal{C}$  is stable under all pullbacks, we have:

### Corollary

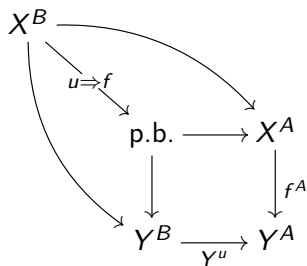
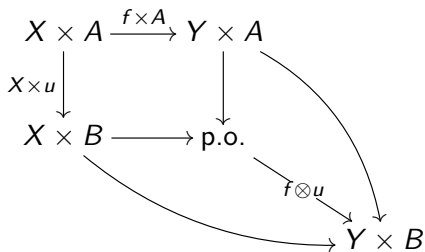
*The trivial fibrations are stable under all pushforwards.*

# The Leibniz adjunction

For any map  $u : A \rightarrow B$  in  $\mathcal{E}$ , the *Leibniz adjunction*

$$(-) \otimes u \dashv u \Rightarrow (-)$$

relates the *pushout-product* with  $u$  and the *pullback-hom* with  $u$ .

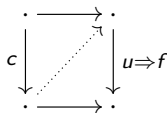
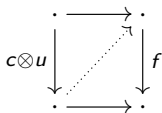


## The Leibniz adjunction

The functors  $(-\otimes u) \dashv (u \Rightarrow -) : \mathcal{E}^2 \longrightarrow \mathcal{E}^2$  also satisfy

$$(c \otimes u) \dashv f \quad \Leftrightarrow \quad c \dashv (u \Rightarrow f)$$

with respect to the diagonal filling relation.



This holds by adjointness.

## The fibration wfs $(\text{TCof}, \mathcal{F})$

We define the fibrations in terms of the trivial fibrations by:

$$f \in \mathcal{F} \quad \text{iff} \quad (\delta \Rightarrow f) \in \text{TFib}$$

where the pullback-hom  $\delta \Rightarrow f$  is with the *generic point*  $\delta : 1 \rightarrow \mathbb{I}$ , in the slice category  $\mathcal{E}/\mathbb{I}$ . Expressed in  $\mathcal{E}$  this gives the following.

### Definition

A map  $f : Y \rightarrow X$  is a *fibration* if  $\delta \Rightarrow f$  below is a trivial fibration.

$$\begin{array}{ccc} Y^{\mathbb{I}} \times \mathbb{I} & \xrightarrow{\text{eval}} & Y \\ \delta \Rightarrow f \downarrow \text{dotted} & \text{p.b.} \longrightarrow & \downarrow f \\ X^{\mathbb{I}} \times \mathbb{I} & \xrightarrow{\text{eval}} & X \\ \uparrow f^{\mathbb{I}} \times \mathbb{I} & & \end{array}$$

## The fibration wfs $(\text{TCof}, \mathcal{F})$

The more familiar “box-filling” condition results by adjointness.

$$\begin{aligned} f \in \mathcal{F} & \text{ iff } (\delta \Rightarrow f) \in \text{TFib} \\ & \text{ iff } c \boxtimes (\delta \Rightarrow f) \text{ for all } c \in \mathcal{C} \\ & \text{ iff } (c \otimes \delta) \boxtimes f \text{ for all } c \in \mathcal{C} \end{aligned}$$

$$\begin{array}{ccc} B \cup (A \times \mathbb{I}) & \longrightarrow & Y \\ \downarrow c \otimes \delta & \nearrow & \downarrow f \\ B \times \mathbb{I} & \longrightarrow & X \end{array}$$



# The fibration wfs $(\text{TCof}, \mathcal{F})$

## Proposition

*There is a wfs  $(\text{TCof}, \mathcal{F})$  with these fibrations as  $\mathcal{F}$ .*

- We define  $\text{TCof}$  by lifting against  $\mathcal{F}$ , so the orthogonality axiom is immediate.
- For the factorization axiom, we use Garner's small object argument.
- Swan has given a constructive version using  $W$ -types.
- The factorization systems  $(\mathcal{C}, \text{TFib})$  and  $(\text{TCof}, \mathcal{F})$  are actually *algebraic*, with associated diagonal filling *structures*.

# Frobenius

Like the trivial fibrations, the fibrations are stable under all pullbacks just because they are a right class. But unlike the trivial fibrations, they are *not* stable under all pushforwards.

However, we do have:

## Theorem

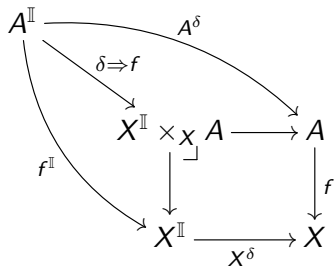
*The fibrations are stable under pushforward along fibrations.*

## Corollary (Frobenius)

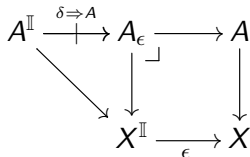
*The trivial cofibrations are stable under pullback along fibrations.*

# Proof

Recall that  $f : A \rightarrow X$  is a fibration iff  $\delta \Rightarrow f$  is a trivial fibration:



We indicate this briefly as follows:



# Proof

Consider fibrations  $B \twoheadrightarrow A \twoheadrightarrow X$ .

Thus we have:

$$\begin{array}{ccccc} B^{\mathbb{I}} & \xrightarrow{\delta \Rightarrow B} & B_{\epsilon}^* & \longrightarrow & B \\ & \searrow & \downarrow \lrcorner & & \downarrow \\ & & A^{\mathbb{I}} & \xrightarrow{\delta \Rightarrow A} & A \\ & & & \searrow & \downarrow \\ & & & & X^{\mathbb{I}} \xrightarrow{\epsilon} X \end{array}$$

# Proof

Taking the pushforward of the right column yields  $\Pi_A B \longrightarrow X$ .

$$\begin{array}{ccccc}
 B^{\mathbb{I}} & \xrightarrow{\delta \Rightarrow B} & B^*_{\epsilon} & \xrightarrow{\quad} & B \\
 & \searrow & \downarrow \lrcorner & & \downarrow \\
 & & A^{\mathbb{I}} & \xrightarrow{\delta \Rightarrow A} & A_{\epsilon} & \xrightarrow{\quad} & A \\
 & & & \searrow & \downarrow \lrcorner & & \downarrow \\
 & & & & X^{\mathbb{I}} & \xrightarrow{\quad} & X \\
 & & & & \uparrow \epsilon & & \uparrow \\
 (\Pi_A B)^{\mathbb{I}} & \xrightarrow{\delta \Rightarrow \Pi_A B} & (\Pi_A B)_{\epsilon} & \xrightarrow{\quad} & \Pi_A B
 \end{array}$$

We want to show that  $\delta \Rightarrow \Pi_A B$  is a trivial fibration.

## Proof

The pushforward of  $\delta \Rightarrow B$  along  $A^{\mathbb{I}} \longrightarrow X^{\mathbb{I}}$  is a trivial fibration over  $X^{\mathbb{I}}$ , since these are closed under all pushforwards.

$$\begin{array}{ccccc}
 B^{\mathbb{I}} & \xrightarrow{\delta \Rightarrow B} & B_{\epsilon}^* & \longrightarrow & B \\
 & \searrow & \downarrow \lrcorner & & \downarrow \\
 & & A^{\mathbb{I}} & \xrightarrow{\delta \Rightarrow A} & A_{\epsilon} & \longrightarrow & A \\
 & & & \searrow & \downarrow \lrcorner & & \downarrow \\
 & & & & X^{\mathbb{I}} & \xrightarrow{\epsilon} & X \\
 & & & & \uparrow & & \uparrow \\
 (\Pi_A B)^{\mathbb{I}} & \xrightarrow{\delta \Rightarrow \Pi_A B} & (\Pi_A B)_{\epsilon} & \longrightarrow & \Pi_A B \\
 & & & & \uparrow & & \uparrow \\
 \Pi_{A^{\mathbb{I}}} B^{\mathbb{I}} & \xrightarrow{\Pi_{A^{\mathbb{I}}} \delta \Rightarrow B} & \Pi_{A^{\mathbb{I}}} B_{\epsilon}^* & & & & 
 \end{array}$$

## Proof

One then shows that there is a retraction of  $\Pi_{A^{\mathbb{I}}}. \delta \Rightarrow B$  onto  $\delta \Rightarrow \Pi_A B$  over  $X^{\mathbb{I}}$ , whence the latter is also a trivial fibration.

$$\begin{array}{ccccc}
 B^{\mathbb{I}} & \xrightarrow{\delta \Rightarrow B} & B_{\epsilon}^* & \xrightarrow{\quad} & B \\
 & \searrow & \downarrow \lrcorner & & \downarrow \\
 & & A^{\mathbb{I}} & \xrightarrow{\delta \Rightarrow A} & A_{\epsilon} & \xrightarrow{\quad} & A \\
 & & & \searrow & \downarrow \lrcorner & & \downarrow \\
 & & & & X^{\mathbb{I}} & \xrightarrow{\epsilon} & X \\
 & & & & \uparrow & & \uparrow \\
 (\Pi_A B)^{\mathbb{I}} & \xrightarrow{\delta \Rightarrow \Pi_A B} & (\Pi_A B)_{\epsilon} & \xrightarrow{\quad} & \Pi_A B \\
 \uparrow \text{dotted} & & \uparrow \text{dotted} & & & & \\
 \Pi_{A^{\mathbb{I}}} B^{\mathbb{I}} & \xrightarrow{\Pi_{A^{\mathbb{I}}}. \delta \Rightarrow B} & \Pi_{A^{\mathbb{I}}} B_{\epsilon}^* & & & & 
 \end{array}$$



## References

For details see:

Awodey: A Quillen model structure on cartesian cubical sets.  
[www.github.com/awodey/math/qms](https://www.github.com/awodey/math/qms) (2019)

This algebraic proof is derived from the type theoretic one in:

Cohen, Coquand, Huber, Mörtberg: Cubical Type Theory:  
A constructive interpretation of the univalence axiom.  
TYPES 2015.



## The retraction

For example to get  $s : (\Pi_A B)_\epsilon \longrightarrow \Pi_{A^\mathbb{I}} B_\epsilon^*$   
 first interpolate  $B_\epsilon$ . Then  $(\Pi_A B)_\epsilon \cong \Pi_{A_\epsilon} B_\epsilon$  by Beck-Chavelley.

$$\begin{array}{ccccccc}
 B^\mathbb{I} & \xrightarrow{\delta \Rightarrow B} & B_\epsilon^* & \cdots & B_\epsilon & \cdots & B \\
 & \searrow & \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \\
 & & A^\mathbb{I} & \xrightarrow{\delta \Rightarrow A} & A_\epsilon & \longrightarrow & A \\
 & & & \searrow & \downarrow \lrcorner & & \downarrow \\
 & & & & X^\mathbb{I} & \xrightarrow{\epsilon} & X \\
 & & & & \uparrow & & \uparrow \\
 (\Pi_A B)^\mathbb{I} & \xrightarrow{\delta \Rightarrow \Pi_A B} & (\Pi_A B)_\epsilon & \longrightarrow & \Pi_A B & & \\
 & & & & \uparrow & & \uparrow \\
 & & & & X^\mathbb{I} & & X \\
 & & & & \downarrow & & \downarrow \\
 & & & & (\Pi_A B)^\mathbb{I} & & (\Pi_A B)_\epsilon \\
 & & & & \downarrow s & & \downarrow \\
 \Pi_{A^\mathbb{I}} B^\mathbb{I} & \xrightarrow{\Pi_{A^\mathbb{I}} \delta \Rightarrow B} & \Pi_{A^\mathbb{I}} B_\epsilon^* & & & & 
 \end{array}$$

