An algebraic proof of the Frobenius condition

Steve Awodey[1]

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Weak factorization systems

Definition

A *weak factorization system* on a category $\mathcal{E}$ consists of two classes of maps $(\mathcal{L}, \mathcal{R})$, both closed under retracts, such that:

(i) Every map $f : A \longrightarrow B$ factors as a map in $\mathcal{L}$ followed by one in $\mathcal{R}$.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, tail, "\in\mathcal{L}"'] & & B \\
& \cdot \arrow[ur, two heads, "\in\mathcal{R}"'] &
\end{tikzcd}$$

(ii) In any commutative square with an $\mathcal{L}$-map on the left and an $\mathcal{R}$-map on the right, there is a diagonal filler making the diagram commute.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail] & X \arrow[d, two heads] \\
B \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

Frobenius

Such structures are part of the definition of a Quillen model category. They are also used to model identity types in HoTT. For that one also needs the following.

Definition

A weak factorization system $(\mathcal{L}, \mathcal{R})$ has the *Frobenius property* if the left maps are stable under pullback along the right maps.

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, tail] \\
X \arrow[r, two heads] & Y
\end{tikzcd}
$$

This condition is equivalent to also modelling Π-types.

Frobenius and Π

The proof uses the following fact about a wfs $(\mathcal{L}, \mathcal{R})$ on an LCC $\mathcal{E}$.

Lemma

For any map $f : Y \longrightarrow X$ with base change $f^* \dashv f_* : \mathcal{E}/Y \longrightarrow \mathcal{E}/X$,

$$f^* \textit{ preserves } \mathcal{L} \quad \textit{iff} \quad f_* \textit{ preserves } \mathcal{R}.$$

Proof.

Suppose f^* preserves $\mathcal{L}$, and let $g : C \longrightarrow D$ in $\mathcal{E}/Y$ be an $\mathcal{R}$-map. Test $f_* g$ against any $c : A \rightarrowtail B$ in $\mathcal{L}$ as on the right below.

$$\begin{tikzcd}
f^*A \arrow[r] \arrow[d, tail, "f^*c"'] & C \arrow[d, two heads, "g"] \\
f^*B \arrow[r] \arrow[ur, dotted] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "c"'] & f_*C \arrow[d, "f_*g"] \\
B \arrow[r] \arrow[ur, dotted] & f_*B
\end{tikzcd}$$

By adjointness and f^* preserves $\mathcal{L}$, we get a diagonal filler as on the left above; thus there is also one on the right. $\square$

Frobenius and Π

Corollary

Suppose $\mathcal{E}$ is LCC and the wfs $(\mathcal{L}, \mathcal{R})$ has the Frobenius property. Then $\mathcal{R}$ is closed under dependent products (Π-types).

$$\begin{tikzcd} A \arrow[d, two heads, "a"'] & \Pi_f A \arrow[d, two heads, "f_* a"] \\ Y \arrow[r, two heads, "f"'] & X \end{tikzcd}$$

Quillen model structures

Definition

A *Quillen model structure* on $\mathcal{E}$ consists of three classes

$$(\mathcal{C}, \mathcal{W}, \mathcal{F})$$

called cofibrations, weak equivalences, and fibrations, such that:

(i) $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are wfs.

(ii) $\mathcal{W}$ has the 3-for-2 property.

Let $\mathrm{TFib} = \mathcal{W} \cap \mathcal{F}$ ("trivial fibrations"),
and $\mathrm{TCof} = \mathcal{C} \cap \mathcal{W}$ ("trivial cofibrations").

QMS and Frobenius

Corollary

Suppose $\mathcal{E}$ is an LCC with QMS $(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

1. *$\mathcal{C}$ is stable under all pullbacks iff*
   TFib *is stable under all pushforwards*
2. $(\mathsf{TCof}, \mathcal{F})$ *has Frobenius iff*
   $\mathcal{F}$ has dependent products.
3. *If $\mathcal{C}$ is stable under pullbacks along $\mathcal{F}$, then:*
   $(\mathsf{TCof}, \mathcal{F})$ *has Frobenius iff*
   $\mathcal{W}$ is stable under pullback along $\mathcal{F}$ ("right proper").

Thus an LCC with a QMS satisfying Frobenius will model MLTT with Σ, Π, Id when taking the fibrations $\mathcal{F}$ as the types.

QMS from a premodel

Definition

A *premodel* in a topos $\mathcal{E}$ consists of $(\Phi, \mathbb{I}, \mathsf{V})$ where:

- $\Phi \hookrightarrow \Omega$ is a representable class of monos (satisfying ...).
- $1 \rightrightarrows \mathbb{I}$ is an interval (satisfying ...).
- $\dot{\mathsf{V}} \to \mathsf{V}$ is a universe (satisfying ...).

Previous work by (A., Coquand, Orton-Pitts, Sattler) provides:

Construction

From a premodel $(\Phi, \mathbb{I}, \mathsf{V})$ *one can construct a QMS* $(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

The **goal for today** is to show:

Theorem

This model structure has Frobenius (and so is right proper).

The cofibration wfs $(\mathcal{C}, \mathrm{TFib})$

The *cofibrations* $\mathcal{C}$ are the monos $C \rightarrowtail Z$ classified by $\Phi \hookrightarrow \Omega$.

$$
\begin{tikzcd}
C \arrow[rr] \arrow[dr, dotted] \arrow[dd, tail] & & 1 \arrow[dd] \\
& 1 \arrow[ur] \arrow[dd, tail] & \\
Z \arrow[rr] \arrow[dr, dotted] & & \Omega \\
& \Phi \arrow[ur, hook] &
\end{tikzcd}
$$

Note that $\mathcal{C}$ is therefore stable under all pullbacks.

The cofibration wfs $(\mathcal{C}, \mathsf{TFib})$

The generic cofibration $1 \rightarrowtail \Phi$ determines a polynomial endofunctor,

$$X^+ := \sum_{\varphi:\Phi} X^{[\varphi]}.$$

This is a (fibered) monad,

$$+ : \mathcal{E} \longrightarrow \mathcal{E}\,,$$

by the *dominance* condition assumed on Φ.

The unit $\eta : X \longrightarrow X^+$ *classifies cofibrant-partial maps* into X.

$$\begin{tikzcd} C \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\eta"] \\ Z \arrow[r, dotted] & X^+ \end{tikzcd}$$

The cofibration wfs $(\mathcal{C}, \mathrm{TFib})$

The *trivial fibrations* are the algebras (A, α) for the pointed endofunctor $+_X : \mathcal{E}/X \longrightarrow \mathcal{E}/X$.

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] & & A^+ \arrow[ll, dotted, bend right, "\alpha"'] \arrow[dl] \\
& X &
\end{tikzcd}$$

Such an algebra (A, α) has lifts against all cofibrations.

$$\begin{tikzcd}
C \arrow[rr] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & & A \arrow[d] \\
Z \arrow[rr, dotted] \arrow[urr, dotted] \arrow[dr] & & A^+ \arrow[u, dotted, bend right, "\alpha"'] \arrow[dl] \\
& X &
\end{tikzcd}$$

The cofibration wfs $(\mathcal{C}, \mathrm{TFib})$

The trivial fibrations form the right class of the *cofibration wfs* $(\mathcal{C}, \mathrm{TFib})$. The factorization axiom follows from the monad multiplication.

Because $\mathcal{C}$ is stable under all pullbacks, we have:

Corollary

The trivial fibrations are stable under all pushforwards.

The Leibniz adjunction

For any map $u : A \to B$ in $\mathcal{E}$, the *Leibniz adjunction*

$$(-)\otimes u \;\dashv\; u\Rightarrow(-)$$

relates the *pushout-product* with u and the *pullback-hom* with u.

$$\begin{tikzcd}
X \times A \arrow[r, "f\times A"] \arrow[d, "X\times u"'] & Y \times A \arrow[d] \arrow[ddr, bend left] & \\
X \times B \arrow[r] \arrow[drr, bend right] & \text{p.o.} \arrow[dr, "f\otimes u"] & \\
 & & Y \times B
\end{tikzcd}
\qquad
\begin{tikzcd}
X^B \arrow[dr, "u\Rightarrow f"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & \text{p.b.} \arrow[r] \arrow[d] & X^A \arrow[d, "f^A"] \\
 & Y^B \arrow[r, "Y^u"'] & Y^A
\end{tikzcd}$$

The Leibniz adjunction

The functors $(-\otimes u) \dashv (u \Rightarrow -) : \mathcal{E}^{\mathbb{2}} \longrightarrow \mathcal{E}^{\mathbb{2}}$ also satisfy

$$(c \otimes u) \boxslash f \quad \Leftrightarrow \quad c \boxslash (u \Rightarrow f)$$

with respect to the diagonal filling relation.

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "c\otimes u"'] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r] \arrow[ur, dotted] & \cdot
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "c"'] & \cdot \arrow[d, "u\Rightarrow f"] \\
\cdot \arrow[r] \arrow[ur, dotted] & \cdot
\end{tikzcd}$$

This holds by adjointness.

The fibration wfs $(\mathsf{TCof}, \mathcal{F})$

We define the fibrations in terms of the trivial fibrations by:

$$f \in \mathcal{F} \quad \text{iff} \quad (\delta \Rightarrow f) \in \mathsf{TFib}$$

where the pullback-hom $\delta \Rightarrow f$ is with the *generic point* $\delta : 1 \to \mathbb{I}$, in the slice category $\mathcal{E}/\mathbb{I}$. Expressed in $\mathcal{E}$ this gives the following.

Definition

A map $f : Y \to X$ is a *fibration* if $\delta \Rightarrow f$ below is a trivial fibration.

$$\begin{tikzcd}
Y^{\mathbb{I}} \times \mathbb{I} \arrow[dr, dotted, "\delta \Rightarrow f"] \arrow[drr, bend left, "\mathrm{eval}"] \arrow[ddr, bend right, "f^{\mathbb{I}} \times \mathbb{I}"'] & & \\
& \mathrm{p.b.} \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
& X^{\mathbb{I}} \times \mathbb{I} \arrow[r, "\mathrm{eval}"'] & X
\end{tikzcd}$$

The fibration wfs $(\mathsf{TCof}, \mathcal{F})$

The more familiar “box-filling” condition results by adjointness.

$$
\begin{aligned}
f \in \mathcal{F} \quad & \text{iff} \quad (\delta \Rightarrow f) \in \mathsf{TFib} \\
& \text{iff} \quad c \boxslash (\delta \Rightarrow f) \quad \text{for all } c \in \mathcal{C} \\
& \text{iff} \quad (c \otimes \delta) \boxslash f \quad \text{for all } c \in \mathcal{C}
\end{aligned}
$$

$$
\begin{tikzcd}
B \cup (A \times \mathbb{I}) \arrow[r] \arrow[d, tail, "c\otimes\delta"'] & Y \arrow[d, "f"] \\
B \times \mathbb{I} \arrow[r] \arrow[ur, dotted] & X
\end{tikzcd}
$$

The fibration wfs $(\mathsf{TCof}, \mathcal{F})$

Proposition

There is a wfs $(\mathsf{TCof}, \mathcal{F})$ *with these fibrations as* $\mathcal{F}$*.*

- We define TCof by lifting against $\mathcal{F}$, so the orthogonality axiom is immediate.
- For the factorization axiom, we use Garner's small object argument.
- Swan has given a constructive version using W-types.
- The factorization systems $(\mathcal{C}, \mathsf{TFib})$ and $(\mathsf{TCof}, \mathcal{F})$ are actually *algebraic*, with associated diagonal filling *structures*.

Frobenius

Like the trivial fibrations, the fibrations are stable under all pullbacks just because they are a right class. But unlike the trivial fibrations, they are *not* stable under all pushforwards.

However, we do have:

Theorem

The fibrations are stable under pushforward along fibrations.

Corollary (Frobenius)

The trivial cofibrations are stable under pullback along fibrations.

Proof

Recall that $f : A \to X$ is a fibration iff $\delta \Rightarrow f$ is a trivial fibration:

$$
\begin{tikzcd}
A^{\mathbb{I}} \arrow[dr, "\delta\Rightarrow f"] \arrow[drr, bend left, "A^{\delta}"] \arrow[ddr, bend right, "f^{\mathbb{I}}"'] & & \\
& X^{\mathbb{I}} \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& X^{\mathbb{I}} \arrow[r, "X^{\delta}"'] & X
\end{tikzcd}
$$

We indicate this briefly as follows:

$$
\begin{tikzcd}
A^{\mathbb{I}} \arrow[r, "\delta\Rightarrow A", "|" marking] \arrow[dr] & A_{\epsilon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& X^{\mathbb{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd}
$$

Proof

Consider fibrations $B \twoheadrightarrow A \twoheadrightarrow X$.

Thus we have:

$$
\begin{tikzcd}
B^{\mathbb{I}} \arrow[r, "\delta\Rightarrow B", "|" marking] \arrow[dr] & B^*_\epsilon \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[d] \\
& A^{\mathbb{I}} \arrow[r, "\delta\Rightarrow A", "|" marking] \arrow[dr] & A_\epsilon \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathbb{I}} \arrow[r, "\epsilon"'] & X
\end{tikzcd}
$$

Proof

Taking the pushforward of the right column yields $\Pi_A B \longrightarrow X$.

$$
\begin{tikzcd}
B^{\mathbb{I}} \arrow[r, "\delta \Rightarrow B", "\shortmid" marking] \arrow[dr] & B^*_\epsilon \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[d] \\
& A^{\mathbb{I}} \arrow[r, "\delta \Rightarrow A", "\shortmid" marking] \arrow[dr] & A_\epsilon \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathbb{I}} \arrow[r, "\epsilon"'] & X \\
& (\Pi_A B)^{\mathbb{I}} \arrow[r, "\delta \Rightarrow \Pi_A B"'] \arrow[ur] & (\Pi_A B)_\epsilon \arrow[r] \arrow[u] & \Pi_A B \arrow[u]
\end{tikzcd}
$$

We want to show that $\delta \Rightarrow \Pi_A B$ is a trivial fibration.

Proof

The pushforward of $\delta \Rightarrow B$ along $A^{\mathbb{I}} \longrightarrow X^{\mathbb{I}}$ is a trivial fibration over $X^{\mathbb{I}}$, since these are closed under all pushforwards.

$$
\begin{tikzcd}
B^{\mathbb{I}} \arrow[r, "\delta\Rightarrow B", "|"{marking}] \arrow[dr] & B^*_\epsilon \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[d] \\
& A^{\mathbb{I}} \arrow[r, "\delta\Rightarrow A", "|"{marking}] \arrow[dr] & A_\epsilon \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathbb{I}} \arrow[r, "\epsilon"'] & X \\
& (\Pi_A B)^{\mathbb{I}} \arrow[r, "\delta\Rightarrow\Pi_A B"'] \arrow[ur] & (\Pi_A B)_\epsilon \arrow[r] \arrow[u] & \Pi_A B \arrow[u] \\
& \Pi_{A^{\mathbb{I}}} B^{\mathbb{I}} \arrow[r, "\Pi_{A^{\mathbb{I}}}.\delta\Rightarrow B"', "|"{marking}] & \Pi_{A^{\mathbb{I}}} B^*_\epsilon \arrow[uu, dotted, bend right] &
\end{tikzcd}
$$

Proof

One then shows that there is a retraction of $\Pi_{A^{\mathbb{I}}}.\delta \Rightarrow B$ onto $\delta \Rightarrow \Pi_A B$ over $X^{\mathbb{I}}$, whence the latter is also a trivial fibration.

$$
\begin{tikzcd}
B^{\mathbb{I}} \arrow[r, "\delta\Rightarrow B", "\shortmid" marking] \arrow[dr] & B^*_{\epsilon} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[d] \\
& A^{\mathbb{I}} \arrow[r, "\delta\Rightarrow A", "\shortmid" marking] \arrow[dr] & A_{\epsilon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathbb{I}} \arrow[r, "\epsilon"'] & X \\
& (\Pi_A B)^{\mathbb{I}} \arrow[r, "\delta\Rightarrow\Pi_A B"'] \arrow[ur] \arrow[d, tail, dotted] & (\Pi_A B)_{\epsilon} \arrow[r] \arrow[u] \arrow[d, tail, dotted] & \Pi_A B \arrow[u] \\
& \Pi_{A^{\mathbb{I}}} B^{\mathbb{I}} \arrow[r, "\Pi_{A^{\mathbb{I}}}.\delta\Rightarrow B"', "\shortmid" marking] \arrow[u, bend left, two heads, dotted] & \Pi_{A^{\mathbb{I}}} B^*_{\epsilon} \arrow[u, bend right, two heads, dotted] &
\end{tikzcd}
$$

$\square$

References

For details see:

Awodey: A Quillen model structure on cartesian cubical sets.
www.github.com/awodey/math/qms (2019)

This algebraic proof is derived from the type theoretic one in:

Cohen, Coquand, Huber, Mörtberg: Cubical Type Theory:
A constructive interpretation of the univalence axiom.
TYPES 2015.

The retraction

For example to get $s : (\Pi_A B)_\epsilon \longrightarrow \Pi_{A^{\mathbb{I}}} B^*_\epsilon$
first interpolate B_ϵ. Then $(\Pi_A B)_\epsilon \cong \Pi_{A_\epsilon} B_\epsilon$ by Beck-Chavelley.

$$
\begin{tikzcd}
B^{\mathbb{I}} \arrow[r, "\delta\Rightarrow B", "\shortmid" marking] \arrow[dr] & B^*_\epsilon \arrow[r, dotted] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_\epsilon \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
& A^{\mathbb{I}} \arrow[r, "\delta\Rightarrow A", "\shortmid" marking] \arrow[dr] & A_\epsilon \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
& & X^{\mathbb{I}} \arrow[r, "\epsilon"'] & X \\
& (\Pi_A B)^{\mathbb{I}} \arrow[ur] \arrow[r, "\delta\Rightarrow\Pi_A B"'] & (\Pi_A B)_\epsilon \arrow[u] \arrow[r] \arrow[d, dotted, "s"] & \Pi_A B \arrow[u] \\
& \Pi_{A^{\mathbb{I}}} B^{\mathbb{I}} \arrow[r, "\Pi_{A^{\mathbb{I}}}\cdot\delta\Rightarrow B"', "\shortmid" marking] & \Pi_{A^{\mathbb{I}}} B^*_\epsilon &
\end{tikzcd}
$$

The retraction

Then use $\Pi_{A^{\mathbb{I}}} B_\epsilon^* \cong \Pi_{A_\epsilon}(B_\epsilon^*)_*$ to get:

$$
\begin{tikzcd}
 & & (B_\epsilon^*)_* & \\
B^{\mathbb{I}} \arrow[r, "\delta\Rightarrow B", "|" marking] \arrow[dr] & B_\epsilon^* \arrow[r, dotted] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_\epsilon \arrow[r, dotted] \arrow[d, dotted] \arrow[u, dotted, "\eta"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
 & A^{\mathbb{I}} \arrow[r, "\delta\Rightarrow A", "|" marking] \arrow[dr] & A_\epsilon \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
 & & X^{\mathbb{I}} \arrow[r, "\epsilon"'] & X \\
 & (\Pi_A B)^{\mathbb{I}} \arrow[r, "\delta\Rightarrow\Pi_A B"'] \arrow[ur] & (\Pi_A B)_\epsilon \arrow[r] \arrow[u] \arrow[d, dotted, "s:=\Pi_{A_\epsilon}\eta"] & \Pi_A B \arrow[u] \\
 & \Pi_{A^{\mathbb{I}}} B^{\mathbb{I}} \arrow[r, "\Pi_{A^{\mathbb{I}}}.\delta\Rightarrow B"', "|" marking] & \Pi_{A^{\mathbb{I}}} B_\epsilon^* &
\end{tikzcd}
$$