An extension of Morley's categoricity theorem to infinite quantifier languages

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However, it is categorical in every λ with respect to the notion of internal size $|A|$ defined as follows. If $r(A)$ is the least regular cardinal λ such that A is λ-presentable, then:

$$|A| = \begin{cases} \kappa & \text{if } r(A) = \kappa^+ \\ r(A) & \text{if } r(A) \text{ is limit} \end{cases}$$

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The construction proceeds in the following steps:

- Take the initial segment of the ordinals up to μ. The natural (Hessenberg) sum and product is defined setting $a + b$ (resp. $a.b$) as the maximum order type of a linear order extending the partial order given by the disjoint union (resp. the direct product).

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- Take the μ-completion of that field considering all μ-Cauchy μ-sequences of fractions.

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Assuming *GCH*, we have:

$$\lambda^\mu = \begin{cases} \lambda & \text{if } \mathit{cof}(\lambda) > \mu \text{ and } 2^\mu < \lambda \\ \lambda^+ & \text{if } \mathit{cof}(\lambda) \leq \mu \text{ and } 2^\mu < \lambda \end{cases}$$

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As a result, eventually there is exactly one μ-Hilbert space of cardinality λ regular not a successor of a cardinal of cofinality $\leq \mu$, but there are two μ-Hilbert spaces (of internal sizes λ and λ^+) if it is such a successor.On the other hand, it is categorical in every λ with respect to internal size.

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Recall that the κ^+-classifying topos of $\mathbb{T}_{\kappa^+}$ (Espindola 2017), $\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}$ is defined through the following universal property:

$$\begin{tikzcd} \mathcal{C}_{\mathbb{T}_{\kappa^+}} \arrow[rr] \arrow[dr] & & \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \arrow[dl, dashed, "\kappa^+-\text{small limit preserving}"] \\ & \mathcal{E} & \end{tikzcd}$$

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Theorem

Let $\mathbb{T}_\kappa$ axiomatize $\mathcal{K}_{\geq\kappa}$. Then the λ-classifying topos of $\mathbb{T}_\kappa$ is equivalent to the presheaf topos $\mathcal{S}et^{Mod_\lambda(\mathbb{T}_\kappa)}$ where $Mod_\lambda(\mathbb{T}_\kappa)$ is the subcategory of λ-presentable models of $\mathbb{T}_\kappa$. Moreover, the canonical embedding of the syntactic category is given by (note that $\mathcal{K}_\kappa \ni M : \mathcal{C}_{\mathbb{T}_\kappa} \to \mathcal{S}et$):

$$\mathcal{C}_{\mathbb{T}_\kappa} \xrightarrow{\quad ev \quad} \mathcal{S}et^{\mathcal{K}_\kappa}$$

$$X \longmapsto M \mapsto M(X)$$

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A model M is μ^+-saturated if for every morphism $N \to N'$ between models of size μ, every morphism $N \to M$ can be extended:

$$\begin{tikzcd} N' \arrow[dr, dashed] & \\ N \arrow[u] \arrow[r] & M \end{tikzcd}$$

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Theorem

The category of κ^+-saturated models $Sat_{\kappa^+}(\mathcal{K})$ is axiomatizable in $\mathcal{L}_{\kappa^{++},\kappa^+}$ and if τ_D is the dense (alternatively, atomic) Grothendieck topology on $\mathcal{K}_\kappa^{op}$ (where $\mathcal{K}_\kappa$ is the subcategory of objects of internal size κ), we have

$$\mathcal{S}et[\mathbb{T}_{\kappa^+}^{sat}]_{\kappa^+} \cong \mathcal{S}h(\mathcal{K}_\kappa^{op}, \tau_D)$$

Large cardinals and amalgamation

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Theorem

Let κ be a strongly compact cardinal and consider an accessible category $\mathcal{K}$ equivalent to the category of models of some $\mathcal{L}_{\kappa,\kappa}$ theory $\mathbb{T}$. If $\mathcal{K}$ is categorical at $\lambda \geq \kappa$, then $\mathcal{K}_{\geq \kappa}$ has the amalgamation property.

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Theorem

Let κ be a strongly compact cardinal and consider an accessible category $\mathcal{K}$ equivalent to the category of models of some $\mathcal{L}_{\kappa,\kappa}$ theory categorical in $\lambda \geq \kappa$. Then any two λ^+-saturated models of size λ^+ are isomorphic.

Eventual categoricity

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Theorem

(Shelah's eventual categoricity conjecture for accessible categories with directed colimits). Assume GCH and that there is a proper class of strongly compact cardinals. Let $\mathcal{K}$ be an accessible category with directed colimits. Then there exists a cardinal μ_0 such that if $\mathcal{K}$ is categorical in some $\lambda \geq \mu_0$, it is categorical in all $\lambda' \geq \mu_0$.

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Proof.

$$
\begin{tikzcd}
\mathcal{S}et^{\mathcal{K}_\kappa} \arrow[d] \arrow[dr, "M"] & \\
\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \arrow[r] \arrow[d] & \mathcal{S}et \\
\mathcal{S}h(\mathcal{K}_\kappa^{op}, \tau_D) \arrow[ur, dashed] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{S}et^{\mathcal{K}_{\geq \kappa^+, \leq \lambda}} \\
\mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \arrow[u, "ev"']
\end{tikzcd}
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□

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Lemma

Let κ be regular, not a successor of a cardinal of small cofinality, and assume $(\kappa^+)^\kappa = \kappa^+$. Then $\mathcal{K}_\kappa$ has the amalgamation property if and only if $\mathcal{S}et^{\mathcal{K}_\kappa}$ is a De Morgan topos (it satisfies $\top \vdash_{\mathbf{x}} \neg\phi \vee \neg\neg\phi$ for κ^+-coherent ϕ).

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Assume GCH. If $\mathcal{K}$ is categorical in both κ and κ^+, $\mathcal{K}_\kappa$ satisfies the amalgamation property.

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$$\begin{tikzcd}
 & \mathcal{S}et[\mathbb{T}_\kappa]_{\kappa^+} \cong \mathcal{S}et^{\mathcal{K}_\kappa} \arrow[d] \\
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