Constructivism and weak logical principles over arithmetic

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We study the interrelation between syntactical restrictions of the following logical principles over (many-sorted) intuitionistic arithmetic.

- LEM (Law of Excluded Middle): $\varphi \vee \neg\varphi$
- DML (De Morgan's Law): $\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$
- DNE (Double Negation Elimination): $\neg\neg\varphi \to \varphi$
- DNS (Double Negation Shift): $\forall x \neg\neg\varphi(x) \to \neg\neg\forall x \varphi(x)$

Motivations

1. One Conceptual Motivation:
   Foundation (Axiomatization) of Constructivism
2. One Practical Motivation:
   Framework for Constructive Reverse Mathematics
3. Applications to other fields (Proof mining, Limit computable math., Computable math. etc.)
4. . . .

Motivation 1: Foundation of Constructivism

- In early 20 centuries, L. E. J. Brouwer tried to reconstruct mathematics in favor of his intuitionism.
- In Brouwer's intuitionistic mathematics (which is a first school of constructive mathematics), everything has to be built from the ground up, including the meaning of the logical symbols.
- A. Heyting, a pupil of Brouwer, formalized Brouwer's "proofs as constructions"-concept via his theory Heyting arithmetic (HA) with an informal semantics (BHK interpretation, below).
- HA is a variant of Peano arithmetic (PA) based on intuitionistic logic, and one can obtain PA just by adding LEM ($\varphi \vee \neg\varphi$) into the axioms of HA.

BHK(Brouwer/Heyting/Kolmogorov)-interpretation

- There is no proof for $\bot$.
- A proof of $A \wedge B$ is a pair (q, r) of proofs, where q is a proof of A and r is a proof of B.
- A proof of $A \vee B$ is a pair (n, q) consisting of an integer n and a proof q, where q is a proof of A if $n = 0$ and q is a proof of B if $n \neq 0$.
- A proof p of $\exists x A(x)$ is a pair (c_d, q), where c_d is the construction of an element d of the domain and q is a proof of $A(d)$.
- A proof p of $A \to B$ is a construction which transforms any proof of A into a proof of B.
- A proof p of $\forall x A(x)$ is a construction which transforms any construction of d of the domain into a proof of $A(d)$.

- The reasoning in constructive/intuitionistic mathematics is somewhat restricted than that in classical (usual) mathematics.

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- Then classical logic is **too strong** as a formalization of constructive reasoning:

Typical Example. A classical rule RAA(reductio ad absurdum)

$$\dfrac{\dfrac{\begin{array}{c}\cancel{\neg S} \\ \hline \vdots\end{array}}{\bot}}{S}\ \mathrm{RAA}$$

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A simple answer is intuitionistic logic.

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- The large amount of Bishop's constructive mathematics can be formalized in a second-order theory EL, or even EL_0 which has Σ^0_1 induction only and is often served as a base theory for constructive reverse mathematics.
- From a foundational point of view, however, a determinative answer on suitable underlying logic for constructive mathematics is still missing (in my thought).

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In the case of arithmetic:

- Some weak logical principles (like $\Delta_{\mathrm{a}}\text{-LEM}$ below) are modified realizable (and also Dialectica interpretable) in Gödel's **T** but NOT provable in HA$^\omega$.
- DNS is modified realizable in **T** (while DNS itself is used for the verification) but NOT provable HA$^\omega$.

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Question. How much α capture constructive provability?

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If we assume Γ as axioms, then we can constructively prove a statement S.	$\approx_?$	$\Gamma \vdash_{i+\alpha} \mathrm{S}$

Motivation 2: Framework for Constructive Reverse Mathematics

- The aim of reverse mathematics is classifying mathematical theorems from a perspective of logical complexity.

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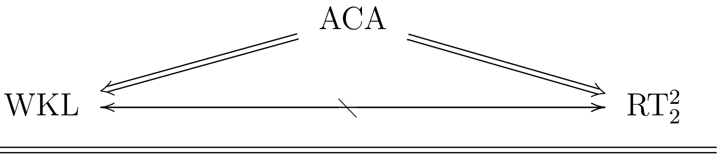
- In reverse mathematics, one formalizes mathematical statements in second-order arithmetic, and investigates the relation between such statements and logical axioms.

Classical Reverse Mathematics (1970's–)

- Many of ordinary (non-set theoretic) mathematical theorems are provable in RCA_0, or equivalent to WKL or ACA over RCA_0. (Friedman, Simpson etc.)
- A well-known exception is Ramsey's theorem for pairs RT^2_2. (Cholak/Jockusch/Slaman 2001, Liu 2012)



RCA_0 (or RCA)

$\approx$ Classical Computable Math.

WKL: Weak König's Lemma, ACA: Arithmetical Comprehension.

Constructive Reverse Mathematics (2000's–)

- Constructive reverse mathematics (Ishihara, Nemoto, Berger etc.), which is reverse mathematics over **intuitionistic** fragment EL_0 (resp. EL) of RCA_0 (resp. RCA)
- In contrast to other attempts in reverse mathematics, its underlying logic is NOT classical logic.

Constructive Reverse Mathematics (2000's–)

- Constructive reverse mathematics (Ishihara, Nemoto, Berger etc.), which is reverse mathematics over **intuitionistic** fragment EL_0 (resp. EL) of RCA_0 (resp. RCA)
- In contrast to other attempts in reverse mathematics, its underlying logic is NOT classical logic.
- One can obtain a much sharper classification rather than classical reverse mathematics:

	Intuitionistic Logic		Classical Logic	
First-order	HA		PA	
Second-order	EL_0	EL	RCA_0	RCA

$$\mathrm{EL}_0, \mathrm{EL} \approx \text{(Bishop's) Constructive Math.}$$

Examples of Constructive Reverse Mathematics.

1. $\mathrm{UC_c}$: Every continuous mapping from $2^{\mathbb{N}}$ to $\mathbb{N}$ with a continuous modulus is uniformly continuous.
   - $\mathsf{RCA}_0 \vdash \mathrm{WKL} \leftrightarrow \mathrm{UC_c}$.
   - $\mathsf{EL}_0 \vdash \mathrm{FAN_D} \leftrightarrow \mathrm{UC_c}$ and $\mathsf{EL}_0 + \mathrm{FAN_D} \nvdash \mathrm{WKL}$.
2. IVT: Intermediate Value Theorem
   - $\mathsf{RCA}_0 \vdash \mathrm{IVT}$.
   - $\mathsf{EL}_0 \vdash \mathrm{IVT} \leftrightarrow \mathrm{WKL^c}$ and $\mathsf{EL}_0 \nvdash \mathrm{IVT}$.

- WKL: Every infinite binary tree has a infinite path.
- $\mathrm{FAN_D}$: Contrapositive of WKL.
- $\mathrm{WKL^c}$: Every infinite binary **convex** tree has a infinite path.

Proposition. (Ishihara 2005)

- $\mathsf{EL}_0 \vdash \mathrm{ACA} \leftrightarrow \Sigma^0_1\text{-LEM} + \Pi^0_1\text{-AC}^{0,0}$.
- $\mathsf{EL}_0 \vdash \mathrm{WKL} \leftrightarrow \Sigma^0_1\text{-DML} + \Pi^0_1\text{-AC}^{\vee}_0$.

- Σ^0_1-LEM: $\forall\alpha\forall x(\exists y\, \alpha(x,y)=0 \vee \neg\exists y\, \alpha(x,y)=0)$.
- Σ^0_1-DML:

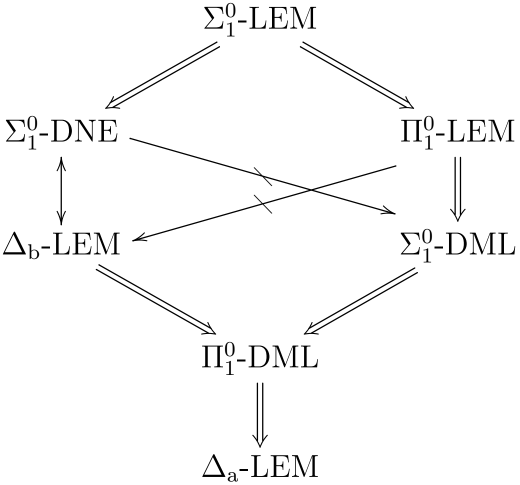
$$\forall\alpha,\beta\forall x\left(\begin{array}{l}\neg(\exists y\,\alpha(x,y)=0 \wedge \exists z\,\beta(x,z)=0)\\ \to \neg\exists y\,\alpha(x,y)=0 \vee \neg\exists z\,\beta(x,z)=0\end{array}\right).$$

Notation: We use $x, y, z \dots$ for number variables and $\alpha, \beta, \gamma \dots$ for function variables.

Remark.

$\mathsf{RCA}_0 = \mathsf{EL}_0 + \mathrm{LEM} = \mathsf{EL}_0 + \mathrm{DNE} \supset \mathsf{EL}_0 + \mathrm{DML}$.

Structure of the weak logical principles over EL_0, EL or HA.



Σ^0_1-LEM: $\forall\alpha\forall x(\exists y\,\alpha(x,y)=0 \lor \neg\exists y\,\alpha(x,y)=0)$.

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Σ^0_1-DML: $\forall\alpha,\beta\forall x(\neg(\exists y\,\alpha(x,y)=0 \land \exists z\,\beta(x,z)=0) \to \neg\exists y\,\alpha(x,y)=0 \lor \neg\exists z\,\beta(x,z)=0)$.

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Σ^0_1-DNE: $\forall\alpha\forall x(\neg\neg\exists y\,\alpha(x,y)=0 \to \exists y\,\alpha(x,y)=0)$.

Δ_a-LEM: $\forall\alpha,\beta\forall x((\exists y\,\alpha(x,y)=0 \leftrightarrow \neg\exists z\,\beta(x,z)=0) \to \exists y\,\alpha(x,y)=0 \lor \neg\exists y\,\alpha(x,y)=0)$.

Δ_b-LEM: $\forall\alpha,\beta\forall x((\neg\exists y\,\alpha(x,y)=0 \leftrightarrow \exists z\,\beta(x,z)=0) \to \exists y\,\alpha(x,y)=0 \lor \neg\exists y\,\alpha(x,y)=0)$.

Realizability and arithmetic in all finite types

Kreisel's **modified realizability** interpretation (1959–) is a kind of formal treatment of the BHK interpretation in the context of arithmetic in all finite types.

Definition. (Modified realizability interpretation)

For a formulas of HA^{ω} (containing only $=_0$ in the language), its realizability interpretation A^{mr} is defined as follows:

- $A^{mr} :\equiv \exists \underline{x}\,(\underline{x} \text{ mr } A) :\equiv A$ for atomic A;

  Let $A^{mr} :\equiv \exists \underline{x}\,(\underline{x} \text{ mr } A)$, $B^{mr} :\equiv \exists \underline{u}\,(\underline{u} \text{ mr } B)$. Then,
- $(A \wedge B)^{mr} :\equiv \exists \underline{x}, \underline{u}\,(\underline{x}, \underline{u} \text{ mr } (A \wedge B))$ $:\equiv \exists \underline{x}, \underline{u}\,(\underline{x} \text{ mr } A \wedge \underline{u} \text{ mr } B)$;
- $(A \vee B)^{mr} :\equiv \exists z^0, \underline{x}, \underline{y}(z, \underline{x}, \underline{y} \text{ mr } (A \vee B))$ $:\equiv \exists z, \underline{x}, \underline{y}\,\big((z =_0 0 \to \underline{x} \text{ mr } A) \wedge (z \neq_0 0 \to \underline{y} \text{ mr } B)\big)$;
- $(A \to B)^{mr} :\equiv \exists \underline{y}\,(\underline{y} \text{ mr } (A \to B))$ $:\equiv \exists \underline{y} \forall \underline{x} (\underline{x} \text{ mr } A \to \underline{y}\underline{x} \text{ mr } B)$;
- $(\forall z^{\rho} A(z))^{mr} :\equiv \exists \underline{x}\,(\underline{x} \text{ mr } \forall z A(z)) :\equiv \exists \underline{x} \forall z (\underline{x}\, z \text{ mr } A(z))$;
- $(\exists z^{\rho} A(z))^{mr} :\equiv \exists z, \underline{x}\,(z, \underline{x} \text{ mr } \exists z A(z)) :\equiv$ $\exists z, \underline{x}\,(\underline{x} \text{ mr } A(z))$.

Proposition. (Soundness of the modified realizability)

If $\mathsf{HA}^\omega + \mathrm{AC}^\omega + \mathrm{IP}^\omega_{\mathrm{ef}} \vdash A$, then there exist terms $\underline{t}$ in Gödel's $\mathbf{T}$ s.t. $\mathsf{HA}^\omega \vdash \underline{t}\ \mathrm{mr}\ A$.

Proposition. (Characterization of the modified realizability)

$\mathsf{HA}^\omega + \mathrm{AC}^\omega + \mathrm{IP}^\omega_{\mathrm{ef}} \vdash A \iff \mathsf{HA}^\omega \vdash A^{mr}$.

- $\mathrm{AC}^{\rho,\tau} : \forall x^\rho \exists y^\tau A(x,y) \to \exists Y^{\tau(\rho)} \forall x^\rho A(x, Y(x))$
(axiom scheme of choice)
- $\mathrm{IP}^\rho_{\mathrm{ef}} : (A_{\mathrm{ef}} \to \exists x^\rho B(x)) \to \exists x^\rho (A_{\mathrm{ef}} \to B(x^\rho))$
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Remark.

Δ_{a}-LEM is provable in $\mathsf{HA}^\omega + \mathrm{IP}^\omega_{\mathrm{ef}}$, but not provable in HA^ω.

Definition. (Dialectica interpretation)

For a formulas of HA^ω (containing only $=_0$ in the language), its Dialectica interpretation A^D is defined as follows:

- $A^D :\equiv A_D :\equiv A\,(\underline{x},\ \underline{y})$ for atomic A;
  Let $A^D \equiv \exists \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y})$, $B^D :\equiv \exists \underline{u} \forall \underline{v} B_D(\underline{u}, \underline{v})$. Then,
- $(A \wedge B)^D :\equiv \exists \underline{x}, \underline{u} \forall \underline{y}, \underline{v}\, (A \wedge B)_D :\equiv \exists \underline{x}, \underline{u} \forall \underline{y}, \underline{v} (A_D(\underline{x}, \underline{y}) \wedge B_D(\underline{u}, \underline{v}))$;
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- $(\forall z^\rho A(z))^D :\equiv \exists \underline{X} \forall z, \underline{y}\, (\forall z A(z))_D :\equiv \exists \underline{X} \forall z, \underline{y} A_D(\underline{X}\, z, \underline{y}, z)$;
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Proposition. (Soundness of the Dialectica interpretation)

If $\mathsf{HA}^\omega + \mathrm{AC}^\omega + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega \vdash A$, then there exist terms $\underline{t}$ in Gödel's **T** s.t. $\mathsf{HA}^\omega \vdash \forall \underline{y} A_D(\underline{t}, \underline{y})$.

Proposition. (Characterization of the Dialectica interpretation)

$\mathsf{HA}^\omega + \mathrm{AC}^\omega + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega \vdash A \iff \mathsf{HA}^\omega \vdash A^D$.

- $\mathrm{IP}^{\rho,\tau}_\forall : (\forall u^\tau A_{qf}(u) \to \exists x^\rho B(x)) \to \exists x^\rho (\forall u^\tau A_{qf}(u) \to B(x))$
  (independence-of-premise schema for purely univ. formulas)
- $\mathrm{M}^\rho : \neg\neg\exists x^\rho A_{qf}(x) \to \exists x^\rho A_{qf}(x)$
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Remark.

M^ω is a finite type extension of Σ^0_1-DNE, and hence the above system contains the weak logical principles below Σ^0_1-DNE.

Unification (Oliva 2006)

Definition. (Modified realizability in the Dialectica style)

- $A^R :\equiv A_R :\equiv A$ for atomic A;

  Let $A^R \equiv \exists \underline{x} \forall \underline{y} A_R(\underline{x}, \underline{y})$, $B^R :\equiv \exists \underline{u} \forall \underline{v} B_R(\underline{u}, \underline{v})$. Then,
- $(A \wedge B)^R :\equiv \exists \underline{x}, \underline{u} \forall \underline{y}, \underline{v}\, (A \wedge B)_R$
  $:\equiv \exists \underline{x}, \underline{u} \forall \underline{y}, \underline{v} (A_R(\underline{x}, \underline{y}) \wedge B_R(\underline{u}, \underline{v}))$;
- $(A \vee B)^R :\equiv \exists z^0, \underline{x}, \underline{u} \forall \underline{y}, \underline{v}\, (A \vee B)_R$
  $:\equiv \exists z^0, \underline{x}, \underline{u} \forall \underline{y}, \underline{v} \big((z = 0 \to A_R(\underline{x}, \underline{y})) \wedge (z \neq 0 \to B_R(\underline{u}, \underline{v})) \big)$;
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  $:\equiv \exists \underline{U} \forall \underline{x} \big(\forall \underline{y}\, A_R(\underline{x}, \underline{y}) \to \forall \underline{v}\, B_R(\underline{U}\, \underline{x}, \underline{v}) \big)$;
- $(\forall z^\rho A(z))^R :\equiv \exists \underline{X} \forall z, \underline{y}\, (\forall z A(z))_R :\equiv \exists \underline{X} \forall z, \underline{y} A_R(\underline{X}\, z, \underline{y}, z)$;
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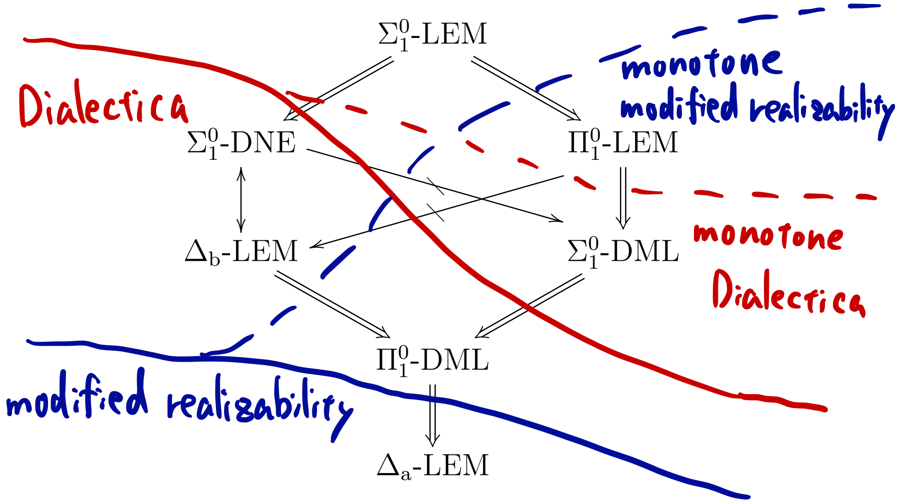
Remark. One can show that A_R is $\exists$-free.

Proposition.

$\mathsf{HA}^\omega \vdash \forall \underline{x} \left(\underline{x} \ \mathrm{mr} \ C \leftrightarrow \forall \underline{y} \, C_R(\underline{x}, \underline{y}) \right)$.

Thus, one can see the modified realizability as a kind of Dialectica (Oliva 2006), and vice versa!

Modified realizability and logical principles



On the status of 2 important restrictions of DNS

- $\mathrm{DNS}: \forall x \neg\neg\varphi(x) \to \neg\neg\forall x \varphi(x)$ is a principle which is modified realizable with an empty realizer.
- For the verification, however, one need DNS itself (as well as AC^{ω}).

Gödel-Spector's Reduction of Consistency

1. In his 1958 paper, Gödel gave a consistency proof of PA by using the Dialectica interpretation.
2. In his 1962 paper, Spector extended Gödel's method to second-order arithmetic (classical analysis).

Their method consists of the following steps:

- Firstly, by negative translation, one reduces a classical theory to the corresponding (semi-)intuitionistic theory,
- Secondly, by the Dialectica interpretation, one reduces the (semi-)intuitionistic theory to the quantifier-free functional theory $\mathbf{T}$ ($+$ bar recursion).

$$\mathsf{PA}^{(\omega)} \vdash \bot \underset{\text{Negative translation}}{\Longrightarrow} \mathsf{HA}^{(\omega)} \vdash \bot^{\mathrm{N}} (\equiv \bot) \underset{\text{Dialectica interpretation}}{\Longrightarrow} \mathbf{T} \models \bot .$$

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and $\mathsf{RCA}_0 + \mathrm{ACA}$ is equivalent to RCA_0 augmented with

$$\Pi^0_1\text{-}\mathrm{AC}^{0,0} : \forall\alpha(\forall x \exists y \forall z\, \alpha(x,y,z) = 0 \to \exists\gamma\forall x, z\, \alpha(x,\gamma(x),z) = 0).$$

Negative Translations of RCA_0 and ACA_0.

- $\mathsf{RCA}_0 \vdash \varphi \Longrightarrow \mathsf{EL}_0 + \Sigma^0_1\text{-DNS}^0 \vdash \varphi^{\mathrm{N}}$.
- $\mathsf{RCA}_0 + \Pi^0_1\text{-AC}^{0,0} \vdash \varphi \Rightarrow \mathsf{EL}_0 + \Pi^0_1\text{-AC}^{0,0} + \Sigma^0_2\text{-DNS}^0 \vdash \varphi^{\mathrm{N}}$.

$\Sigma^0_1\text{-DNS}^0$: $\forall\alpha(\forall x \neg\neg\exists y\, \alpha(x,y) = 0 \rightarrow \neg\neg\forall x \exists y\, \alpha(x,y) = 0)$.

$\Sigma^0_2\text{-DNS}^0$:

$\forall\alpha(\forall x \neg\neg\exists y \forall z\, \alpha(x,y,z) = 0 \rightarrow \neg\neg\forall x \exists y \forall z\, \alpha(x,y,z) = 0)$.

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- $\mathsf{RCA}_0 \vdash \varphi \Longrightarrow \mathsf{EL}_0 + \Sigma^0_1\text{-DNS}^0 \vdash \varphi^{\mathrm{N}}$.
- $\mathsf{RCA}_0 + \Pi^0_1\text{-AC}^{0,0} \vdash \varphi \Rightarrow \mathsf{EL}_0 + \Pi^0_1\text{-AC}^{0,0} + \Sigma^0_2\text{-DNS}^0 \vdash \varphi^{\mathrm{N}}$.

$\Sigma^0_1\text{-DNS}^0$: $\forall \alpha (\forall x \neg\neg \exists y\, \alpha(x,y) = 0 \to \neg\neg \forall x \exists y\, \alpha(x,y) = 0)$.

$\Sigma^0_2\text{-DNS}^0$:

$\forall \alpha (\forall x \neg\neg \exists y \forall z\, \alpha(x,y,z) = 0 \to \neg\neg \forall x \exists y \forall z\, \alpha(x,y,z) = 0)$.

Observation.

1. $\mathsf{EL}_0 + \mathrm{AC}^{0,0} + \Sigma^0_1\text{-DNE}$ (which derives $\Sigma^0_1\text{-DNS}^0$) is Dialectica interpretable in $\mathbf{T}$.
2. $\Sigma^0_2\text{-DNS}^0$ is NOT Dialectica interpretable in $\mathbf{T}$ (while it is Dialectica interpretable in $\mathbf{T} + \mathbf{BR}_{0,1}$).

Observation from Pure (Predicate) Logic

Fact 1.

$\mathbf{CQC} = \mathbf{IQC} + \mathrm{LEM} = \mathbf{IQC} + \mathrm{DML} + \mathrm{DNE}.$

Let us consider about $\neg\neg\mathrm{LEM}$.

Fact 2.

- $\mathbf{IQC} \vdash \forall x \neg\neg(\varphi(x) \vee \neg\varphi(x))$.
- $\mathbf{IQC} \nvdash \neg\neg\forall x(\varphi(x) \vee \neg\varphi(x))$.

Fact 3.

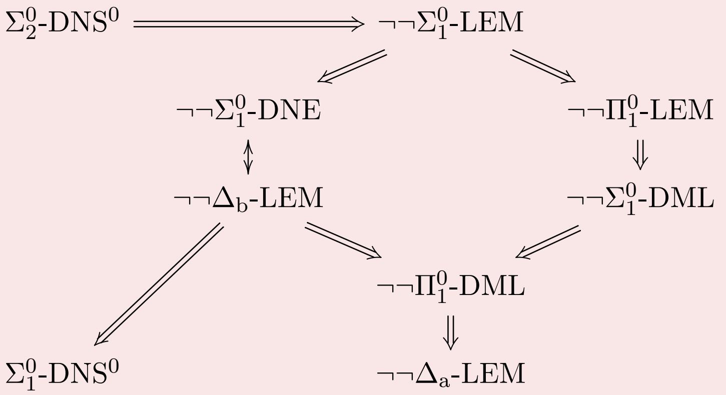
$\mathbf{IQC} \vdash \mathrm{DNS} \leftrightarrow \neg\neg\mathrm{LEM}.$

- $\mathrm{DNS}: \forall x \neg\neg\varphi(x) \to \neg\neg\forall x \varphi(x)$.
- $\neg\neg\mathrm{LEM}: \neg\neg\forall x(\varphi(x) \vee \neg\varphi(x))$.

The proof shows $\Sigma^0_2\text{-}\mathrm{DNS}^0 \to \neg\neg\Sigma^0_1\text{-}\mathrm{LEM} \to \Sigma^0_1\text{-}\mathrm{DNS}^0$.

Results (Kohlenbach/F. 2018)

Whenever an implication does not follow by transitivity, then it actually does not hold!



EL_0, EL or HA

Warning

The hierarchy **above** $\neg\neg\Sigma^0_1\text{-LEM}$ collapses in the presence of the axiom of choice!

Fact.

For all *n*, $\mathsf{EL}_0 + \Pi^0_1\text{-AC}^{0,0} + \neg\neg\Sigma^0_1\text{-LEM} \vdash \neg\neg\Sigma^0_n\text{-LEM}$.

Proposition. (Kohlenbach 2015)

- Π^0_1-DML is NOT modified realizable in **T**.
- Δ_a-LEM is modified realizable in **T**.

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Lemma. (Ishihara/Nemoto/F. 2015)

EL $+ \mathrm{ECT}_0 + \Delta_a$-LEM $\vdash \Pi^0_1$-DML, where ECT_0 is the following (classically false but intuitionistically consistent) statement:

$\forall x(A \to \exists y B(x,y)) \to \exists z \forall x(A \to \exists u(T(z,x,u) \wedge B(x,U(u)))$,

where A is almost negative.

Proposition. (Kohlenbach 2015)

- Π^0_1-DML is NOT modified realizable in **T**.
- Δ_a-LEM is modified realizable in **T**.

Lemma. (Ishihara/Nemoto/F. 2015)

EL $+ \mathrm{ECT}_0 + \Delta_a$-LEM $\vdash \Pi^0_1$-DML, where ECT_0 is the following (classically false but intuitionistically consistent) statement:

$$\forall x(A \to \exists y B(x,y)) \to \exists z \forall x(A \to \exists u(T(z,x,u) \wedge B(x,U(u))),$$

where A is almost negative. In the same manner, we have EL $+ \mathrm{ECT}_0 + \neg\neg\Delta_a$-LEM $\vdash \neg\neg\Pi^0_1$-DML.

Proposition. (Kohlenbach 2015)

- Π^0_1-DML is NOT modified realizable in **T**.
- Δ_a-LEM is modified realizable in **T**.

Lemma. (Ishihara/Nemoto/F. 2015)

$\mathsf{EL} + \mathrm{ECT}_0 + \Delta_a\text{-LEM} \vdash \Pi^0_1\text{-DML}$, where ECT_0 is the following (classically false but intuitionistically consistent) statement:

$$\forall x(A \to \exists y B(x,y)) \to \exists z \forall x(A \to \exists u(T(z,x,u) \wedge B(x,U(u)))),$$

where A is almost negative. In the same manner, we have $\mathsf{EL} + \mathrm{ECT}_0 + \neg\neg\Delta_a\text{-LEM} \vdash \neg\neg\Pi^0_1\text{-DML}$.

Theorem.

$\mathsf{EL} + \mathrm{ECT}_0 + \Sigma^0_1\text{-DNS}^0 \nvdash \neg\neg\Delta_a\text{-LEM}$.

Proof Sketch.

Suppose that EL $+ \mathrm{ECT}_0 + \Sigma^0_1\text{-}\mathrm{DNS}^0 \vdash \neg\neg\Delta_{\mathrm{a}}\text{-}\mathrm{LEM}$.
By the previous lemma, we have that EL $+ \mathrm{ECT}_0 + \Sigma^0_1\text{-}\mathrm{DNS}^0$ (equivalently $\Sigma^0_1\text{-}\mathrm{DNS}^{0-}$) proves $\neg\neg\Pi^0_1\text{-}\mathrm{DML}$, in particular,

$$\neg\neg\forall x \left(\begin{array}{l} \neg(\forall y\ s(x,y) = 0 \wedge \forall z\ t(x,z) = 0) \\ \to \neg\forall y\ s(x,y) = 0 \vee \neg\forall z\ t(x,z) = 0 \end{array} \right) \qquad (1)$$

where s, t are from Kohlenbach 2015.
Then HA $+ \mathrm{ECT}_0 + \Sigma^0_1\text{-}\mathrm{DNS}^{0-} \vdash (1)$.
By Kleene realizability, one can drop ECT_0 and obtain
$\mathrm{HA}^\omega + \mathrm{AC} + \Sigma^0_1\text{-}\mathrm{DNS}^0$ proves

$$\neg\neg\exists f\forall x \left(\begin{array}{l} \neg(\forall y\ s(x,y) = 0 \wedge \forall z\ t(x,z) = 0) \\ \to \left(\begin{array}{l} (f(x) = 0 \to \neg\forall y\ s(x,y) = 0) \\ \wedge (f(x) \neq 0 \to \neg\forall z\ t(x,z) = 0 \end{array} \right) \end{array} \right). \qquad (2)$$

By modified realizability, we have
$\mathrm{HA}^\omega + \mathrm{QF\text{-}AC}^{0,0} + \Sigma^0_1\text{-}\mathrm{DNS}^0 \vdash (2)$.
HRO is a model of the former but (2) is false in HRO. □

Corollary

$\mathsf{EL} + \mathrm{AC}^{0,0} + \Sigma^0_1\text{-}\mathrm{DNS}^0 \nvdash \neg\neg\Delta_{\mathrm{a}}\text{-}\mathrm{LEM}$
and $\mathsf{HA} + \Sigma^0_1\text{-}\mathrm{DNS}^0 \nvdash \neg\neg\Delta_{\mathrm{a}}\text{-}\mathrm{LEM}$ in the language of HA.

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A Possible Direction

An interesting direction concerning foundations of constructivism is to investigate the weak logical principles in other theories with respect to constructivism (as Martin-Löf type theory).

"Neutral" constructivism and the logical principles

- The logical principles weaker than Markov's principle (Σ^0_1-DNE) are accepted both in classical mathematics and Russian constructive recursive mathematics.
- While Markov's principle is not accepted in intuitionistic mathematics, it is consistent with e.g. $\mathsf{HA}^\omega + \mathrm{AC}^\omega + \text{C-N} + \mathrm{BI_M}$ (Ishihara/Nemoto).
- Then it is interesting to see whether some weak logical principles are provable in (the system corresponding to) intuitionistic mathematics (without Kripke scheme).

Proposition.

Σ^0_2-DNS^0 is equivalent to $(\neg\neg\Sigma^0_1)$-$\mathrm{BI_M}$ over $\mathsf{EL}_0 + \Pi^0_1\text{-}\mathrm{AC}^{0,0} + \Sigma^0_1\text{-}\mathrm{DNS}^1$, where

$$\Sigma^0_1\text{-}\mathrm{DNS}^1 : \forall\alpha\neg\neg\exists x A_{\mathrm{qf}}(\alpha, x) \to \neg\neg\forall\alpha\exists x A_{\mathrm{qf}}(\alpha, x).$$

Thank you for your attention!