Dependently typed theories as generalised Lawvere theories

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Memorial conference for Erik Palmgren

19 Nov. 2020

Multisorted algebraic theories

An **algebraic theory** (or **Lawvere theory**) consists of :

1. A set S of sorts,
2. a set $\mathcal{F}$ of sorted function symbols, each written

$$x{:}A_1, \ldots x_n{:}A_n \vdash f : A \qquad (A_1, \ldots, A_n, A \in S).$$

3. A set of equations between terms over $\mathcal{F}$.

Dependently sorted algebraic theories

Question

What should a **dependently** sorted algebraic theory be?

Many approaches

- Cartmell's generalised algebraic theories [Car78].
- Makkai's logic with dependent sorts [Mak95].
- Fiore's Σ_n-models with substitution [Fio08].
- Palmgren's DFOL signatures [Pal16].
- Others (Aczel, Belo, QIITs ...)

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…but which has the « *bon goût* » of closely resembling algebraic/Lawvere theories from a category-theoretic point of view.

In fact, what I'll call **dependently typed theories** are exactly the Σ_0-models with substitution of [Fio08]. (I wish I had known this earlier.)

Equivalent categorical definitions of algebraic theory

Let $S \in \mathcal{S}\mathrm{et}$.

Definition

An S-sorted algebraic theory is a category with finite products whose objects are freely generated by S.

Definition

An S-sorted algebraic theory is a finitary monad on $\mathcal{S}\mathrm{et}/S = \widehat{S}$.

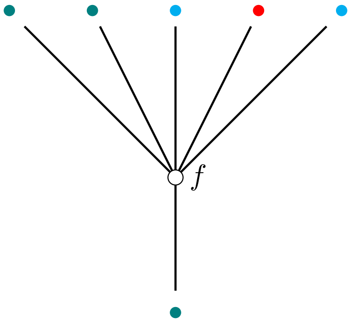
Let $\mathcal{F}\mathrm{in}(S) = \mathcal{F}\mathrm{in}/S$ be the (small) category of finite sets over S. Then the presheaf category $\mathcal{S}\mathrm{et}^{\mathcal{F}\mathrm{in}(S)\times S}$ of **cartesian S-sorted term signatures** has a "substitution" monoidal product.

Definition

An S-sorted algebraic theory is a monoid in the monoidal category $\mathcal{S}\mathrm{et}^{\mathcal{F}\mathrm{in}(S)\times S}$.

Combinatorics

These definitions are based on a combinatorial view of substitution of sorted terms of an algebraic theory as "cartesian" grafting of trees (cartesian = with weakening and duplication of inputs).



A term in a multisorted Lawvere theory takes a finite coproduct of sorts as input, and has an output sort.

Whence S-sorted term signatures as objects of $\mathcal{S}\mathrm{et}^{\mathcal{F}\mathrm{in}(S)\times S}$.

Generalising this picture to dependent types

Dependent type signatures

We begin with a syntactic definition.

A **(dependent type) signature** S is a graded set $S = \coprod_{n \in \mathbb{N}} S_n$, such that each S_j is a set of *type declarations* over the signature $S_{<j} = \coprod_{i<j} S_i$.

A **type declaration** over a signature S is a pair (Γ, A) where Γ is a (Martin-Löf) context typed by S and A is a (fresh) type symbol.

Examples

-

$$\vdash \mathrm{Ob}\ \textsf{type}$$

$$x, y{:}\mathrm{Ob} \vdash \mathrm{Hom}(x, y)\ \textsf{type}$$

-

$$\vdash C\ \textsf{type}$$

$$\forall k \in \mathbb{N}, \quad x_1, \ldots, x_k, y{:}C \vdash O(x_1, \ldots, x_k, y)\ \textsf{type}$$

Every signature S has a **syntactic category** whose objects are contexts Γ typed by S and whose morphisms are context morphisms $\Gamma \to \Delta$. Since there are are no term symbols, all morphisms are substitutions of variables only.

For a signature S, let C_S be the full subcategory of its category of contexts on the contexts $(\Gamma, x{:}A)$ for all (Γ, A) in S.

Categorical parenthesis

Definition

A **direct** category is a small category C such that the relation

$$c < d \qquad \Leftrightarrow \qquad \exists \text{ a non-identity arrow } c \to d$$

on the objects of C is well-founded (i.e. no infinite chains $\ldots < c_0$).

Definition

A category C is **locally finite** if each of its slice categories is finite.

Categorical definition of a signature

Signatures are precisely locally finite direct categories (cf. [Mak95, Fio08]).

Proposition

The map $S \mapsto C_S^{\mathrm{op}}$ is an equivalence between signatures and locally finite direct categories.

Examples of dependent type signatures

0. Any set S, seen as a discrete category.
1. The category $\{s, t : 0 \rightrightarrows 1\}$.
2. The category $\mathbb{G}$ of globes.
3. The category Δ_+ of semi-simplices.
4. The category $\mathrm{elt}_{\mathrm{pl}}$ of planar corollas/elementary trees.
5. The category Ω_{pl} of planar trees.
6. The category $\mathbb{O}$ of opetopes.

Dependently typed theories

Syntactically, a **dependently typed theory** consists of:

1. A dependent type signature S,
2. an ordered set of *term declarations* of the form $\Gamma \vdash f : A\sigma$,
3. and an ordered set of equations of the form $\Gamma \vdash t_1 = t_2 : A\sigma$,

where Γ is any context typed by S, σ is a term substitution, and t_1 and t_2 are typed terms.

Example of a dependently typed theory

The theory of **categories**:

$$\begin{aligned} &\vdash \mathrm{Ob}\ \mathsf{type} \\ x, y{:}\mathrm{Ob} &\vdash \mathrm{Hom}(x, y)\ \mathsf{type} \\ x{:}\mathrm{Ob} &\vdash 1_x : \mathrm{Hom}(x, x) \\ \ldots, a{:}\mathrm{Hom}(x, y), b{:}\mathrm{Hom}(y, z) &\vdash b \circ a : \mathrm{Hom}(x, z) \\ x, y{:}\mathrm{Ob}, a{:}\mathrm{Hom}(x, y) &\vdash a \circ 1_x = a : \mathrm{Hom}(x, y) \\ x, y{:}\mathrm{Ob}, a{:}\mathrm{Hom}(x, y) &\vdash 1_y \circ a = a : \mathrm{Hom}(x, y) \\ \ldots &\vdash (c \circ b) \circ a = c \circ (b \circ a) : \mathrm{Hom}(x_1, x_4) \end{aligned}$$

We will see that there are dependently typed theories of 2-categories, n-categories, ω-categories, reflexive graphs, simplicial sets, opetopic sets, planar operads ...

Categorical definition

Let $\mathbb{S}$ be a locally finite direct category (let S be the corresponding signature). Let $\mathcal{F}\mathrm{in}(\mathbb{S})$ denote the full subcategory of $\widehat{\mathbb{S}}$ of the finitely presentable objects.

Recall that any X in $\widehat{\mathbb{S}}$ is in $\mathcal{F}\mathrm{in}(\mathbb{S})$ just when X is a finite colimit of representables.

For each s in $\mathbb{S}$, let $\mathbb{S}^{-}_{/s}$ denote the full subcategory of the slice category $\mathbb{S}/s$ such that the only object not in $\mathbb{S}^{-}_{/s}$ is the identity morphism $1_s : s \to s$. The colimit of the functor $\mathbb{S}^{-}_{/s} \to \mathbb{S} \hookrightarrow \widehat{\mathbb{S}}$ is a subobject $\partial s \hookrightarrow s$ called the **boundary** of the representable presheaf s.

Since $\mathbb{S}$ is locally finite, ∂s is finitely presentable for every s in $\mathbb{S}$.

Definition

A **finite cell complex** is a finite sequence of morphisms $\emptyset \to X_0 \ldots \to X_n$ in $\widehat{\mathbb{S}}$ where each morphism $X_i \to X_{i+1}$ is a *chosen* pushout of some $\partial s \hookrightarrow s$.

Lemma

Any X in $\widehat{\mathbb{S}}$ is finitely presentable if and only if there exists a finite cell complex $\emptyset \to \ldots \to X$.

We define $\mathcal{C}\mathrm{ell}(\mathbb{S})$ to be the category whose objects are finite cell complexes, and such that $\mathrm{Hom}(\emptyset \ldots \to X, \emptyset \ldots \to Y) = \widehat{\mathbb{S}}(X, Y)$.

Clearly, the functor $\mathcal{C}\mathrm{ell}(\mathbb{S}) \to \mathcal{F}\mathrm{in}(\mathbb{S})$ is an equivalence of categories.

Proposition

$\mathcal{C}\mathrm{ell}(\mathbb{S})^{\mathrm{op}}$ is *isomorphic* to the syntactic category of the signature S.

Corollary

$\mathcal{C}\mathrm{ell}(\mathbb{S})^{\mathrm{op}}$ is a contextual category.

Definition

The category $\mathcal{C}\mathrm{oll}_{\mathbb{S}}$ of **cartesian $\mathbb{S}$-sorted term signatures** is defined to be the presheaf category $\left[\mathcal{C}\mathrm{ell}(\mathbb{S}), \widehat{\mathbb{S}}\right]$.

For every context Γ typed by S and every type declaration s in S, a term signature X in $\mathcal{C}\mathrm{oll}_{\mathbb{S}}$ gives (functorially) a set $(X\Gamma)_s$ of *term declarations* of type s in the context Γ.

Categorical parenthesis

Let C be a small category and let $\mathcal{F}\mathrm{in}(C)$ be as previously.

The presheaf category $\left[\mathcal{F}\mathrm{in}(C), \widehat{C}\right]$ has a "substitution" monoidal product defined by

$$((Y \circ X)\Gamma)_c := \int^{\Theta \in \mathcal{F}\mathrm{in}(C)} (Y\Theta)_c \times \widehat{C}(\Theta, X\Gamma)$$

whose unit is the inclusion functor $E : \mathcal{F}\mathrm{in}(C) \hookrightarrow \widehat{C}$.

The functor $\mathrm{Lan}_E(-) : \left[\mathcal{F}\mathrm{in}(C), \widehat{C}\right] \to \left[\widehat{C}, \widehat{C}\right]$ of left Kan extension along $E : \mathcal{F}\mathrm{in}(C) \hookrightarrow \widehat{C}$ is (1) **fully faithful** and (2) **monoidal**.

$$\begin{tikzcd}
\mathcal{F}\mathrm{in}(C) \arrow[r, "X"] \arrow[d, hook, "E"'] \arrow[dr, phantom, "\cong"] & \widehat{C} \\
\widehat{C} \arrow[ur, "\mathrm{Lan}_E X"'] &
\end{tikzcd} \tag{1}$$

$$\mathrm{Lan}_E(Y \circ X) \;\cong\; \mathrm{Lan}_E Y \circ \mathrm{Lan}_E X \quad ; \quad \mathrm{Lan}_E E \;\cong\; \mathrm{id}_{\widehat{C}} \tag{2}$$

Proposition

There is an equivalence of categories between monoids in $\left[\mathcal{F}\mathrm{in}(C), \widehat{C}\right]$ and finitary monads on $\widehat{C}$.

Equivalent definitions of dependently typed theory

From the previous parenthesis, we have a substitution monoidal product on $\mathcal{C}\mathrm{oll}_{\mathbb{S}}$.

The **term algebra** of $X \in \mathcal{C}\mathrm{oll}_{\mathbb{S}}$ is the free monoid on X.

Definition

An $\mathbb{S}$-sorted **dependently typed theory** is a monoid in $\mathcal{C}\mathrm{oll}_{\mathbb{S}} \simeq \left[\mathcal{F}\mathrm{in}(\mathbb{S}), \widehat{\mathbb{S}}\right]$.

Definition

An $\mathbb{S}$-sorted **dependently typed theory** is a finitary monad on $\widehat{\mathbb{S}}$.

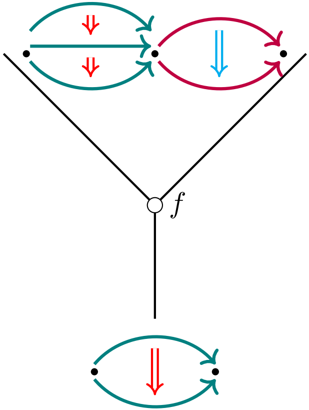
Definition

An $\mathbb{S}$-sorted **dependently typed theory** is an $\mathbb{S}$-contextual category.

The last definition generalises the "finite-product category" definition of algebraic theories.

Combinatorics of dependently typed theories

The substitution monoidal product for $\mathbb{S}$-sorted terms can also be seen as "cartesian" grafting of trees.



A term of an $\mathbb{S}$-sorted dependently typed theory takes a finite cell complex as input, and has as output sort a cell (i.e. an object of $\mathbb{S}$).

(This point of view is closely related to Burroni-Leinster T-operads.)

Examples of dependently typed theories

0. Every multisorted algebraic theory.
1. The identity monads on $\mathcal{G}\mathrm{ph}, \widehat{\mathbb{G}}, \widehat{\Delta_+}, \widehat{\mathrm{elt}_{\mathrm{pl}}}, \widehat{\Omega_{\mathrm{pl}}}, \widehat{\mathbb{O}}$.
2. The free-category monad on $\mathcal{G}\mathrm{ph}$.
3. The free-planar (coloured) operad monad on $\widehat{\mathrm{elt}_{\mathrm{pl}}}$.
4. The free simplicial set monad on semi-simplicial sets.
5. The free-strict-ω-category monad on $\widehat{\mathbb{G}}$.
6. The free-weak-ω-category monad on $\widehat{\mathbb{G}}$.
7. For $T : \widehat{\mathbb{S}} \to \widehat{\mathbb{S}}$ a finitary cartesian monad, every T-operad (à la Burroni-Leinster).

Theorem (L.S., LeFanu Lumsdaine)

The following categories are equivalent:

1. *The category* $\mathcal{C}\mathrm{xlCat}(\mathbb{S})$ *of* $\mathbb{S}$*-contextual categories.*
2. *The category* $\mathcal{M}\mathrm{on}(\mathcal{C}\mathrm{oll}_{\mathbb{S}}, \circ, E)$ *of monoids in cartesian* $\mathbb{S}$*-sorted term signatures.*
3. *The category of finitary monads on* $\widehat{\mathbb{S}}$*.*

3'. *The category of Lawvere theories with arities* $\mathcal{C}\mathrm{ell}(\mathbb{S}) \to \widehat{\mathbb{S}}$*.*

Conclusion

In sum,

- We introduce **dependently typed theories** as a generalisation of multisorted algebraic theories.
- These "cartesian dependent multicategories" are less expressive than many other syntactic approaches, but have a nice algebraic description.
- They manage to capture a large number of well-known examples.

Reflections on dependently coloured operads

Regular algebraic theories

A term $\Gamma \vdash t : A$ of a multisorted algebraic theory is **linear** (or "planar") if each variable in Γ appears exactly once in t, and in the same order as in Γ.
A multisorted algebraic theory is **strongly regular** if each of its equations is between "linear" terms.

Strongly regular algebraic theories and planar coloured operads are closely related.

Coloured operads

Let S be a set of sorts ("colours" in operad jargon). Then the free monoidal category on S is (equivalent to) the set ΣS of finite lists of elements of S. There is an obvious surjective on objects functor $\Sigma S \to \mathcal{F}\mathrm{in}(S)$ taking $(s_1, \ldots, s_k)$ to the coproduct of the representables $s_1, \ldots, s_k$.

The category of **linear** S-sorted term signatures is the presheaf category $\mathcal{S}\mathrm{et}/(\Sigma S \times S) = \left[\Sigma S, \widehat{S}\right]$.

The linear substitution monoidal product on $\left[\Sigma S, \widehat{S}\right]$ is given by **convolution** :

First, for $X \in \left[\Sigma S, \widehat{S}\right]$ and $(s_1, \ldots, s_k) \in \Sigma S$ we define $X^{(s_1,\ldots,s_k)} \in [\Sigma S, \mathcal{S}\mathrm{et}]$ as the Day convolution

$$(X_{s_1} \otimes \ldots \otimes X_{s_k})_{(s'_1,\ldots,s'_m)} := \sum_{\substack{f:\{s_1,\ldots,s_k\}\to\Sigma S \\ fs_1+\ldots+fs_k=(s'_1,\ldots,s'_m)}} \prod_{i=1}^{k} X(fs_i)_{s_i}.$$

$(X^{(s_1,\ldots,s_k)})_{(s'_1,\ldots,s'_m)}$ is the set of *linear* substitutions $(s'_1,\ldots,s'_m) \to (s_1 \ldots, s_k)$ using terms from X.

Next, for $X, Y \in \left[\Sigma S, \widehat{S}\right]$, we define $(Y \circ X) \in \left[\Sigma S, \widehat{S}\right]$ by

$$((Y \circ X)\bar{v})_s := \sum_{\bar{w} \in \Sigma S} (Y\bar{w})_s \times (X^{\bar{w}})_{\bar{v}}$$

This is just the combinatorics of grafting planar labeled trees.

An S-coloured planar operad is a monoid in $\left[\Sigma S, \widehat{S}\right]$.

“Convolution” for $\mathbb{S}$-sorted signatures?

For a dependent type signature $\mathbb{S}$, there seems to be an analogous category $\Sigma\mathbb{S}$ with essentially the same objects as $\mathcal{C}\mathrm{ell}(\mathbb{S})$.

Given an object $X \in \left[\Sigma\mathbb{S}, \widehat{\mathbb{S}}\right]$ and $\Gamma, \Delta \in \mathcal{C}\mathrm{ell}(\mathbb{S})$, we can define the set of linear substitutions $\Delta \to \Gamma$ using terms from X as

$$\sum_{\substack{f:\mathbb{S}/\Gamma \to \Sigma\mathbb{S} \\ \text{“}\mathrm{colim}\text{”} f = \Delta}} \int_{x:s \to \Gamma} X(fx)_s,$$

where the end is over the functor

$$\mathbb{S}/\Gamma \times (\mathbb{S}/\Gamma)^{\mathrm{op}} \xrightarrow{f \times \mathsf{p}} \mathcal{C}\mathrm{ell}(\mathbb{S}) \times \mathbb{S}^{\mathrm{op}} \xrightarrow{X} \mathcal{S}\mathrm{et}.$$

Question

Does this give a monoidal product and a notion of dependently coloured operad?

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